A TREATISE

ON

HIGHER TRIGONOMETRY.

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HIGHER TRIGONOMETRY

BY THE

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PREFACE.

THE present work is intended to complete the subject of Plane Trigonometry as far as it is usually read in Schools and in the junior Classes at the Universities. The introduction of the hyperbolic sine and cosine is an innovation which seems fully justified by their importance in other subjects, and by the simplification effected by their use in the statement of many theorems and formulæ. I wish to thank the Master of Gonville and Caius College for his valuable advice and assistance and for his permission to insert the proof given in Art. 62. The method of proof employed in Art. 47 was first suggested by Professor Adams.

I shall be very grateful for any suggestions or corrections from teachers or students

ETON,
March, 1884.

The Second Edition has been revised and a few alterations have been made where experience suggested that the text was too concise.

The Hyperbolic Functions have been treated more fully on account of their increasing importance in the Integral Calculus.

A method has been inserted of obtaining the expression of $\sin\theta$ and $\cos\theta$ in factors which does not depend on the factors of $x^{2n} - 2x^n \cos\theta - 1$.

The Third Edition has been very carefully revised and many misprints have been corrected. The chapter on the interpretation of $\sqrt{-1}$ has been re-written.

The Fourth Edition has been enlarged by the addition of a chapter on Antiparallels.

J B. L

CAIUS COLLEGE,
April, 1891

NOTE.

References to the Articles in the Higher Trigonometry are given thus [Art. 100]; references to the Elementary Trigonometry thus [E. 100].

The Articles and Exercises which are marked with a star should be omitted when the subject is read for the first time.

The order of the Chapters may in many cases be varied at the discretion of the teacher; in particular the last two Chapters may often be read as an Appendix to the Elementary Trigonometry.

Those of the examples which are not original, have been selected from the various Examination Papers which have been set at Cambridge in the Tripos and in the different College Examinations during the past forty years. Various Examination Papers are appended for the information of intending Candidates

CONTENTS.

CHAPTER I

The Exponential and Logarithmic Series.

1. The **series**

$$1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \frac{1}{|\underline{3}} + \frac{1}{|\underline{4}} + \textbf{etc. ad inf.}$$

is of importance. Hence we prove as follows:

I. **It** is positive, and it is **less than 3.**

For it is less than

$$1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \text{etc.},$$

i.e. less than $\quad 1 + 1 + \frac{1}{2}\left\{\frac{1}{1-\frac{1}{2}}\right\},$

i.e. less than $\quad 1 + 1 + 1.$

II. Since it is positive and less than 3, it is **convergent.**

III. Its value is **2·71828182...**

This may be easily calculated. See Ex. I. (1) page 2.

IV. It is **incommensurable.**

For suppose that it is commensurable; it can then be put into the form $\frac{m}{n}$ where m and n are integers. In this case

$$\frac{m}{n} = 1 + \frac{1}{|\underline{1}} + \frac{1}{|\underline{2}} + \frac{1}{|\underline{3}} + \text{etc.} + \frac{1}{|\underline{n}} + \frac{1}{|\underline{n+1}} + \text{etc.}$$

Multiply each side of this supposed identity by $\lfloor n$, then

$$m \lfloor n-1 = \text{a whole number} + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \text{etc.}$$

But
$$\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \text{etc}$$

is a proper fraction; for it is greater than $\frac{1}{n+1}$ and less than

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \text{etc., } i.e. \text{ less than } \frac{1}{n}.$$

Hence we have to suppose that $m\lfloor n-1$ (a whole number) = a whole number + a proper fraction; which is absurd.

V. Since the numerical value of the series is incommensurable, and we know of no surd or other algebraical expression that is equal to it, it is usual to express its numerical value by the letter e [cf. E. 28]

EXAMPLES. I.

(1) Calculate the value of e by taking the first eleven terms of the series.

(2) Prove that the first 13 terms of the series will give the value of e correct to 9 significant figures.

(3) Prove that $\frac{2}{\lfloor 1} + \frac{4}{\lfloor 3} + \frac{6}{\lfloor 5} + \frac{8}{\lfloor 7} + \text{etc.} = e.$

(4) Prove that $\frac{1}{\lfloor 2} + \frac{1+2}{\lfloor 3} + \frac{1+2+3}{\lfloor 4} + \frac{1+2+3+4}{\lfloor 5} + \text{etc.} = \frac{e}{2}.$

(5) Prove that the series $1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \text{etc}$ is convergent for all values of x.

(6) Prove that the series $x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \text{etc.}$ is convergent if x is greater than -1 and is not greater than 1.

2. **Expansion of e^x in ascending powers of x.**

Since $$\left(1+\frac{1}{n}\right)^{nx}=\left\{\left(1+\frac{1}{n}\right)^n\right\}^x \text{ always } \ldots\ldots\ldots\ldots \text{I.}$$

and since by the binomial theorem

$$\left(1+\frac{1}{n}\right)^{nx}=1+nx\left(\frac{1}{n}\right)+\frac{nx(nx-1)}{1\,.\,2}\left(\frac{1}{n}\right)^2+\text{etc.}$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{1\,.\,2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1\,.\,2\,.\,3}+\text{etc. II.}$$

And similarly

$$\left(1+\frac{1}{n}\right)^n=1+1+\frac{1\left(1-\frac{1}{n}\right)}{1\,.\,2}+\frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1\,.\,2\,.\,3}+\text{etc. III.}$$

Therefore I may be written

$$\left\{1+1+\frac{1\left(1-\frac{1}{n}\right)}{1\,.\,2}+\frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1\,.\,2\,.\,3}+\text{etc.}\right\}^x$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{1\,.\,2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1\,.\,2\,.\,3}+\text{etc}$$

This statement is arithmetically intelligible and true provided both these series are convergent.

They are convergent for all values of n greater than 1.

Therefore they are arithmetically intelligible and true however great n may be. And in the limit, when n is infinitely increased, the above statement becomes [cf. Art. 8]

$$\left\{1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\text{etc.}\right\}^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\text{etc.,}$$

or, $$e^x=1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}+\text{etc.}$$

This result is called the **Exponential Theorem.**

3. To expand a^x in ascending powers of x.

We have $$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \text{etc.}$$

Let a be any number, and let $c = \log_e a$, so that $e^c = a$.

Then $$a^x = (e^c)^x = e^{cx} = e^{x \log_e a}.$$

Therefore
$$a^x = 1 + \frac{x \log_e a}{\lfloor 1} + \frac{x^2 (\log_e a)^2}{\lfloor 2} + \frac{x^3 (\log_e a)^3}{\lfloor 3} + \text{etc.}$$

The Logarithmic Expansion.

4. In the above expansion put $1 + y$ for a, and we obtain
$$(1 + y)^x = 1 + \frac{x \log_e (1 + y)}{\lfloor 1} + \frac{x^2 \{\log_e (1 + y)\}^2}{\lfloor 2} + \text{etc.}$$

This may be put into a different form thus.
$$\frac{(1 + y)^x - 1}{x} = \log_e (1 + y) + \frac{x}{\lfloor 2} \{\log_e (1 + y)\}^2 + \text{terms containing higher powers of } x$$
$$= \log_e (1 + y) + x \,.\, R,$$
where R is a quantity which is not infinite when $x = 0$.

The limit of the right-hand side when $x = 0$ is $\log_e (1 + y)$.

The limit of the left-hand side may be found thus:
$$\frac{(1+y)^x - 1}{x} = \frac{1}{x}\left\{1 + xy + \frac{x(x-1)}{\lfloor 2} y^2 + \frac{x(x-1)(x-2)}{\lfloor 3} y^3 + \text{etc.} - 1\right\}$$
$$= y + \frac{x-1}{\lfloor 2} y^2 + \frac{(x-1)(x-2)}{\lfloor 3} y^3 + \text{etc.},$$
and this, when $x = 0$, has for its limit
$$y - \tfrac{1}{2}y^2 + \tfrac{1}{3}y^3 - \tfrac{1}{4}y^4 + \text{etc.}$$

This series is convergent when y is equal to or numerically less than 1.

Therefore, when y lies between -1 and $+1$ or is equal to 1,

$$\log_e(1+y) = y - \tfrac{1}{2}\cdot y^2 + \tfrac{1}{3}\cdot y^3 - \tfrac{1}{4}\cdot y^4 + \text{etc.}$$

This is the required **Logarithmic Expansion.**

EXAMPLES. II.

(1) Calculate the numerical value of twelve terms of the series $1 + x + \dfrac{x^2}{\underline{|2}} + \dfrac{x^3}{\underline{|3}} + \text{etc.}$ when $x = -1$, and show that it $= \dfrac{1}{e}$.

Prove the following statements

(2) $\dfrac{2}{\underline{|3}} + \dfrac{4}{\underline{|5}} + \dfrac{6}{\underline{|7}} + \text{etc.} = \dfrac{1}{e}$.

(3) $\log_e 2 = \dfrac{1}{2} + \dfrac{1}{3\,.\,4} + \dfrac{1}{5\,.\,6} + \text{etc.} = 1 - \dfrac{1}{2\,.\,3} - \dfrac{1}{4\,.\,5} - \dfrac{1}{6\,.\,7} - \text{etc}$,
and calculate the value of $\log_e 2$ to 2 decimal places. (Result ·69)

(4) $\dfrac{1}{y^2} + \dfrac{1}{2y^4} + \dfrac{1}{3y^6} + \text{etc.} = 2\log_e y - \log_e(y+1) - \log_e(y-1)$.

(5) $\log_e \dfrac{1+y}{1-y} = 2\left\{y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \text{etc}\right\}$.

(6) $2\left\{\dfrac{1}{2y+1} + \dfrac{1}{3(2y+1)^3} + \dfrac{1}{5(2y+1)^5} + \text{etc.}\right\}$

$$= \log_e \frac{1+y}{y} = \log_e(1+y) - \log_e y.$$

(7) $2\log_e y - \log_e(y+1) - \log_e(y-1)$

$$= 2\left\{\frac{1}{2y^2-1} + \frac{1}{3(2y^2-1)^3} + \frac{1}{5(2y^2-1)^5} + \text{etc.}\right\}.$$

(8) Prove that

$$\log_e(n+1) - \log_e(n-1) = 2\left\{\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \text{etc.}\right\}.$$

(9) Use the series of Ex. (8) to prove that $\log_e 3 = 1{\cdot}098612$.

(10) Use the series of Ex. (7) and the result of Ex. (9) to prove that $\log_e 2 = {\cdot}693147$.

(11) Use the result of Ex. (9) and the series of Ex. (6) to prove that $\log_e 10 = 2{\cdot}302585$.

On the Calculation of the Table of Logarithms.

5. The series for $\log_e(1+y)$ is only convergent provided y is not greater than 1 and is greater than -1; also, unless y is small, the series converges very slowly.

It is therefore not a convenient series for the purposes of numerical calculation.

We proceed to obtain such a series.

6. Since $\log_e(1+y) = y - \frac{1}{2}y^2 + \frac{1}{3}y^3 - \frac{1}{4}y^4 + \text{etc.}$ [Art. 4.]
therefore $\log_e(1-y) = -y - \frac{1}{2}y^2 - \frac{1}{3}y^3 - \frac{1}{4}y^4 - \text{etc.}$

Hence by subtraction

$$\log_e \frac{1+y}{1-y} = 2\{y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \text{etc.}\},$$

where y must not be numerically greater than 1.

Let m and n be positive integers, and let m be $> n$; then $\dfrac{m-n}{m+n}$ is less than 1. Put $\dfrac{m-n}{m+n}$ for y in the above result.

Then $$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \text{etc.}\right\}.$$

Let $m = n+1$ in the above; then

$$\log_e(n+1) - \log_e n = 2\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \text{etc.}\right\}.$$

This series is rapidly convergent, and we have thus an easy method for obtaining the logarithms to the base e of *successive* numbers.

Logarithms to the base e are called **Napierian** Logarithms from their inventor. [E. 227.]

The logarithms are calculated thus:

Since $\log_e(n+1)-\log_e n=2\left\{\dfrac{1}{2n+1}+\dfrac{1}{3(2n+1)^3}+\text{etc.}\right\}$,

in this formula put 1 for n.

Then since $\log 1=0$, we can calculate the value of $\log_e 2$.

Next put 2 for n in the above formula, and we can calculate the value of $\log_e 3$.

And so on.

The number of terms of the series which it is necessary to include diminishes as n increases.

In this way a Table of the 'Napierian' Logarithms, of all whole numbers up to any desired magnitude may be determined.

On the Calculation of Common Logarithms.

7. We know that $\log_{10} n = \log_e n \div \log_e 10$. [E. 209, 210.]

Hence, a table of Napierian Logarithms having been constructed, from it we take $\log_e 10\ (=2{\cdot}3025850..)$ and calculate $\dfrac{1}{\log_e 10}={\cdot}43429448$; the table of Common Logs of whole numbers is then formed by multiplying each of the corresponding Napierian Logarithms by ${\cdot}43429448..$.

EXAMPLES. III.

(1) From the preceding data calculate

$$\log_{10} 2,\quad \log_{10} 3,\quad \log_{10} 9.$$

(2) Find $\log_e 7$ and thence calculate $\log_{10} 7$.

(3) If $\mu=\log_{10} e$, prove that

$$\log_{10}(n+1)-\log_{10} n$$

$$=2\mu\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\text{etc.}\right\}.$$

8. In Art. 2 the limit, when n is infinite, of

$$\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\left(x-\frac{3}{n}\right)\cdot\cdot\left(x-\frac{r-1}{n}\right)}{\lfloor r},$$

is assumed to be $\dfrac{x^r}{\lfloor r}$.

This is clear as long as r is not comparable with n.

That it is true for all values of r may be proved by induction, thus:

Assume that the above expression $=\dfrac{x^r}{\lfloor r}+R$, where R is a quantity whose limit is zero when n is infinite; multiply each side by the factor $\left(x-\dfrac{r}{n}\right)\dfrac{1}{r+1}$, and we obtain

$$\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)\cdot\cdot\left(x-\frac{r}{n}\right)}{\lfloor r+1}$$

$$=\frac{x^{r+1}}{\lfloor r+1}+\frac{x\,.\,R}{r+1}-\frac{1}{n}\cdot\frac{r}{r+1}\cdot\left\{\frac{x^r}{\lfloor r}+R\right\}.$$

In this, when $n=\infty$, $R=0$, $\dfrac{1}{n}=0$, and $\dfrac{r}{r+1}\dfrac{x^r}{\lfloor r}$ diminishes as r increases. Thus, if the proposition is true for any value of r it is also true for $r+1$. This proves the proposition.

9. *To prove that the limit of* $\left(\cos\dfrac{\alpha}{n}\right)^n$, *when* n *is infinitely increased is* 1.

Since $$\cos^2\frac{a}{n}=1-\sin^2\frac{a}{n};$$

$$\therefore\ \log\left(\cos\frac{a}{n}\right)^n=\log\left(1-\sin^2\frac{a}{n}\right)^{\frac{n}{2}}=\frac{n}{2}\log\left(1-\sin^2\frac{a}{n}\right)$$

$$=-\frac{n}{2}\left\{\sin^2\frac{a}{n}+\frac{1}{2}\sin^4\frac{a}{n}+\frac{1}{3}\sin^6\frac{a}{n}\text{ etc.}\right\}.\quad\text{[Art. 4.]}$$

The series in the bracket is convergent since $\sin^2\dfrac{a}{n}$ is

less than 1. The series is less than the G. P.

$$\sin^2\frac{a}{n} + \sin^4\frac{a}{n} + \sin^6\frac{a}{n} + \text{etc.},$$

i. e. less than $\dfrac{\sin^2\frac{a}{n}}{1-\sin^2\frac{a}{n}}$, that is, than $\tan^2\frac{a}{n}$,

$$\therefore \log\left(\cos\frac{a}{n}\right)^n \text{ is between } 0 \text{ and } -\frac{1}{2}\left\{n\tan^2\frac{a}{n}\right\},$$

that is between 0 and $-\dfrac{1}{2}\left\{\dfrac{n^2}{a^2}\tan^2\dfrac{a}{n}\right\}\dfrac{a^2}{n}$.

The limit of $\dfrac{n^2}{a^2}\tan^2\dfrac{a}{n}$ when $n = \infty$ is 1 [E. 290.]

The limit of $\dfrac{a^2}{n}$ when $n = \infty$ is 0.

$$\therefore \log\left(\cos\frac{a}{n}\right)^n = 0 \text{ when } n = \infty,$$

and therefore the limit of $\left(\cos\dfrac{a}{n}\right)^n$ is 1. Q. E. D.

10. *The limit of* $\left(\dfrac{\sin\frac{a}{n}}{\frac{a}{n}}\right)^n$ *when* n *is infinitely increased is* 1.

We have (E. 289) 1, $\dfrac{\theta}{\sin\theta}$, $\dfrac{1}{\cos\theta}$ in ascending order of magnitude, when θ is less than 90°. Therefore also

$$(1)^n,\ \left(\frac{\frac{a}{n}}{\sin\frac{a}{n}}\right)^n,\ \frac{1}{\left(\cos\frac{a}{n}\right)^n},$$

are in ascending order of magnitude.

Now let n be infinitely increased and then $\left(\dfrac{\frac{a}{n}}{\sin\frac{a}{n}}\right)^n$ lies between 1 and a quantity whose limit, by Art. 9, is 1. Q.E.D.

*EXAMPLES. IV.

(1) Prove that the limit of $\left(\cos\frac{a}{n}\right)^r$ when n is infinitely increased is 1.

(2) Prove that the limit of $\left(\frac{\sin\theta}{\theta}\right)^r$ when θ is infinitely diminished is 1.

(3) Prove that the limit of $\left(\cos\frac{a}{n}\right)^{n^2}$ is $e^{-\frac{a^2}{2}}$ when n is infinitely increased.

(4) Prove that the limit of $\left(\cos\frac{a}{n}\right)^{n^3}$ is zero when n is infinitely increased.

(5) Prove that the limit of $(\cos\theta)^{\frac{a^m}{\theta^m}}$ where m is an integer, and θ is infinitely diminished is zero, $e^{-\frac{a^2}{2}}$, or 1 according as m is greater, equal to, or less than 2.

*MISCELLANEOUS EXAMPLES. V.

(1) Since $a^x = \{1+(a-1)\}^x$, prove by expanding the right-hand side that
$$a^x = 1 + A_1x + A_2x^2 + A_3x^3 + \text{etc.},$$
where
$$A_1 = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \text{etc.}$$

(2) Since $a^{x+y} = a^x \times a^y$, expand a^{x+y} and a^x by the theorem of Ex. 1, and by equating coefficients of x, prove that
$$A_1 + 2A_2y + 3A_3y^2 + \text{etc.} = A_1a^y.$$

Expand a^y, and by equating the coefficients of the various powers of y find A_2, A_3, etc. in terms of A_1.

Result $$a^x = 1 + A_1x + \frac{A_1^2x^2}{\lfloor 2} + \frac{A_1^3x^3}{\lfloor 3} \text{ etc}$$

(3) Show that $a^{\frac{1}{A_1}}$ in the last example is e.

Hence by Ex. 1 prove that
$$\log_e a = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \text{etc.}$$

(4) Prove that $\log_e n = m\left\{(n^{\frac{1}{m}}-1) - \tfrac{1}{2}(n^{\frac{1}{m}}-1)^2 + \tfrac{1}{3}(n^{\frac{1}{m}}-1)^3 - \text{etc.}\right\}$

Hence, having given that $10^{\frac{1}{2^{32}}} - 1 = \cdot 000000000536112$, prove that
$$\log_e 10 = 2\cdot 30258.$$

(5) Prove that $\log(n+d) - \log n = \mu \left\{ \frac{d}{n} - \frac{d^2}{2n^2} + \frac{d^3}{3n^3} - \text{etc.} \right\}$.

Hence if n be a number greater than 10000 and d a number less than 1, prove that $\frac{\log(n+d) - \log n}{\log(n+d') - \log n} = \frac{d}{d'}$ to a sufficient approximation for all practical purposes.

(6) Prove that

$$\sin\theta = 2^n \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3} \times \ldots \times \cos\frac{\theta}{2^n}\sin\frac{\theta}{2^n},$$

and hence deduce Euler's formula

$$\sin\theta = \theta\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3} \times \qquad \times \cos\frac{\theta}{2^n} \times$$

(7) Prove that

$$\log\left(\frac{\sin\theta}{\theta}\right) = \log\cos\frac{\theta}{2} + \log\cos\frac{\theta}{2^2} + \log\cos\frac{\theta}{2^3} + \text{etc.}$$

(8) Prove that

$$\log\sin 2\alpha + \log\cot\alpha = \cos 2\alpha - \tfrac{1}{2}\cos^2 2\alpha + \tfrac{1}{3}\cos^3 2\alpha - \text{etc.}$$

CHAPTER II.

De Moivre's Theorem.

11. We are about to make use of the symbol $\sqrt{-1}$, which we define to be *an expression whose square is* -1. It is clear that, whatever meaning is given to the symbol $\sqrt{-1}$, it would be quite useless to give to it a meaning such that $\sqrt{-1}$ could not be shewn to follow the ordinary laws of Algebra. In Chapter VII a meaning is given for $\sqrt{-1}$, and it is shewn that with that meaning, $\sqrt{-1}$ does obey the fundamental laws of Algebra.

In the present Chapter we shall use $\sqrt{-1}$ and assume that it does obey the laws of Algebra †.

Since the squares of all numbers whether positive or negative are always positive, and since $(\sqrt{-1})^2 = -1$, it follows that $\sqrt{-1}$ cannot represent any numerical quantity.

Again (by the laws of Algebra)

$$\sqrt{-a^2} = \sqrt{-1 \times a^2} = \sqrt{-1} \times a,$$

therefore $\sqrt{-a^2}$ cannot represent any numerical quantity, accordingly $\sqrt{-1} \times a$ is called an imaginary expression, and it follows that such a statement as $A + B\sqrt{-1} = a + b\sqrt{-1}$ can only be true when $A = a$ and $B = b$.

12. **De Moivre's Theorem.** *Whatever be the value of* n *positive or negative, integral or fractional, cos* nα+$\sqrt{-1}$ *sin* nα *is* one *of the values of* (*cos* α + $\sqrt{-1}$ *sin* α)^n.

I. When n is a **positive integer.**

Consider the product

$$(\cos \alpha + \sqrt{-1} \sin \alpha) \times (\cos \beta + \sqrt{-1} \sin \beta).$$

† Cf C Smith's Algebra, Art. 180.

It is equal to

$$\cos a \,.\, \cos \beta - \sin a \,.\, \sin \beta + \sqrt{-1}\,(\cos a \,.\, \sin \beta + \sin a \,.\, \cos \beta)$$

That is to $\quad \cos(a+\beta) + \sqrt{-1}\,\sin(a+\beta)$.

Similarly the product

$$\{\cos(a+\beta) + \sqrt{-1}\,\sin(a+\beta)\} \times \{\cos\gamma + \sqrt{-1}\,\sin\gamma\}$$

is equal to $\quad \cos(a+\beta+\gamma) + \sqrt{-1}\,\sin(a+\beta+\gamma)$.

Proceeding in this way we obtain that the product of any number n of factors, each of the form $\cos a + \sqrt{-1}\,\sin a$ is equal to

$$\cos(a+\beta+\gamma+\ldots n \text{ terms}) + \sqrt{-1}\,\sin(a+\beta+\gamma+ \quad n \text{ terms}).$$

In this result let $\beta = \gamma = \text{etc.} = a$, and we have that

$$(\cos a + \sqrt{-1}\,\sin a)^n = \cos na + \sqrt{-1}\,\sin na.$$

Thus, when n is a positive integer, De Moivre's Theorem is true.

II. When n is a **negative integer.**

Let $n = -m$. Then m is a positive integer. And

$$(\cos a + \sqrt{-1}\,\sin a)^n = (\cos a + \sqrt{-1}\,\sin a)^{-m}$$

$$= \frac{1}{(\cos a + \sqrt{-1}\,\sin a)^m} = \frac{1}{\cos ma + \sqrt{-1}\,\sin ma} \qquad [\text{By I.}]$$

$$= \frac{1}{\cos ma + \sqrt{-1}\,\sin ma} \times \frac{\cos ma - \sqrt{-1}\,\sin ma}{\cos ma - \sqrt{-1}\,\sin ma}$$

$$= \frac{\cos ma - \sqrt{-1}\,\sin ma}{\cos^2 ma + \sin^2 ma} = \frac{\cos ma - \sqrt{-1}\,\sin ma}{1}.$$

Therefore $(\cos a + \sqrt{-1}\,\sin a)^n = \cos ma - \sqrt{-1}\,\sin ma$

$$= \cos(-m)\,a + \sqrt{-1}\,\sin(-m)\,a = \cos na + \sqrt{-1}\,\sin na.$$

Thus De Moivre's Theorem is true when n is a negative integer.

III. When n is a **fraction**, positive or negative.

Let $n = \frac{p}{q}$, where p and q are integers.

Now $(\cos\beta + \sqrt{-1}\sin\beta)^q = \cos q\beta + \sqrt{-1}\sin q\beta$. [By I. and II.]

Therefore taking the q^{th} root of both sides

$$\cos\beta + \sqrt{-1}\sin\beta$$

is one of the values of $(\cos q\beta + \sqrt{-1}\sin q\beta)^{\frac{1}{q}}$,

or, writing a for $q\beta$, $\cos\frac{a}{q} + \sqrt{-1}\sin\frac{a}{q}$

is one of the values of $(\cos a + \sqrt{-1}\sin a)^{\frac{1}{q}}$.

Therefore $\left(\cos\frac{a}{q} + \sqrt{-1}\sin\frac{a}{q}\right)^p$,

that is $\cos\frac{pa}{q} + \sqrt{-1}\sin\frac{pa}{q}$ [By I.]

is one of the values of $(\cos a + \sqrt{-1}\sin a)^{\frac{p}{q}}$.

Thus the theorem is completely established.

Example 1. *If A stand for* $\cos 2a + i\sin 2a$, *and B, C, D for similar expressions in terms of β, γ, δ, simplify* $(A+B)(C+D)$

$$\begin{aligned}(A+B) &= \cos 2a + i\sin 2a + \cos 2\beta + i\sin 2\beta \\ &= 2\cos(a+\beta)\cos(a-\beta) + 2i\sin(a+\beta)\cos(a-\beta) \\ &= 2\cos(a-\beta)\{\cos(a+\beta) + i\sin(a+\beta)\},\end{aligned}$$

so $(C+D) = 2\cos(\gamma-\delta)\{\cos(\gamma+\delta) + i\sin(\gamma+\delta)\}$,

$$\begin{aligned}\therefore\ (A+B)(C+D) &= 4\cos(a-\beta)\cos(\gamma-\delta)\{\cos(a+\beta)+i\sin(a+\beta)\} \\ &\qquad \{\cos(\gamma+\delta)+i\sin(\gamma+\delta)\} \\ &= 4\cos(a-\beta)\cos(\gamma-\delta\ \{\cos(a+\beta+\gamma+\delta) \\ &\qquad + i\sin(a+\beta+\gamma+\delta)\}\end{aligned}$$

by De Moivre's Theorem.

Example 2. *Assuming* De Moivre's Theorem, *prove that*

$$\sin(\alpha+\beta+\gamma)=\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\gamma\cos\alpha$$
$$+\sin\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta\sin\gamma.$$

By De Moivre's Theorem we have

$$\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)$$
$$=(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)$$
$$=\cos\alpha\cos\beta\cos\gamma-\cos\alpha\sin\beta\sin\gamma-\cos\beta\sin\gamma\sin\alpha$$
$$-\cos\gamma\sin\alpha\sin\beta+i\{\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\gamma\cos\alpha$$
$$+\sin\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta\sin\gamma\}.$$

Now an identity of the form $A+iB=H+iK$ can only be true when $A=H$ and $B=K$; hence

$$\sin(\alpha+\beta+\gamma)=\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\gamma\cos\alpha$$
$$+\sin\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta\sin\gamma.$$

EXAMPLES. VI.

(1) If A stand for $\cos 2\alpha+i\sin 2\alpha$, and B, C, D for similar expressions in terms of β, γ, δ, prove that

$$AB+CD=2\cos(\alpha+\beta-\gamma-\delta)\{\cos(\alpha+\beta+\gamma+\delta)+i\sin(\alpha+\beta+\gamma+\delta)\}.$$

(2) With the notation of Ex. 1, prove that

$$\frac{1}{AB-CD}=\frac{\sin(\alpha+\beta+\gamma+\delta)+i\cos(\alpha+\beta+\gamma+\delta)}{2\sin(\alpha+\beta-\gamma-\delta)}$$

(3) With the same notation, prove that

$$(A-B)(C-D)$$
$$=-4\sin(\alpha-\beta)\sin(\gamma-\delta)\{\cos(\alpha+\beta+\gamma+\delta)+i\sin(\alpha+\beta+\gamma+\delta)\}.$$

(4) With the same notation, prove that

$$\frac{1}{(A+B)(C+D)}=\frac{\cos(\alpha+\beta+\gamma+\delta)-i\sin(\alpha+\beta+\gamma+\delta)}{4\cos(\alpha-\beta)\cos(\gamma-\delta)}.$$

(5) With the same notation, prove that if

$$\frac{1}{(A-B)(C-D)}=H+iK,$$

then $$H=\frac{1}{4}\cos(\alpha+\beta+\gamma+\delta)\operatorname{cosec}(\beta-\alpha)\operatorname{cosec}(\gamma-\delta).$$

(6) Assuming De Moivre's Theorem, prove that

$$\cos(\alpha+\beta-\gamma)=\cos\alpha\cos\beta\cos\gamma+\cos\alpha\sin\beta\sin\gamma+\cos\beta\sin\gamma\sin\alpha$$
$$-\cos\gamma\sin\alpha\sin\beta.$$

(7) By the aid of De Moivre's Theorem express $\sin(\alpha+\beta+\gamma+\delta)$ and $\cos(\alpha+\beta+\gamma+\delta)$ in terms of the sines and cosines of α, β, γ and δ.

(8) By the aid of De Moivre's Theorem express $\sin 4\alpha$ and $\cos 4\alpha$ in terms of the sines and cosines of α.

(9) Prove that

$$\cos(\alpha+\beta+\gamma \ldots n \text{ terms}) + i \sin(\alpha+\beta+\gamma+ \quad n \text{ terms})$$
$$= \cos\alpha \,.\, \cos\beta \,.\, \cos\gamma \;.\, n \text{ factors}$$
$$\{(1+i\tan\alpha)(1+i\tan\beta)(1+i\tan\gamma) \qquad n \text{ factors}\}$$

(10) Prove that

$$\frac{\sin(\alpha+\beta+\gamma+\ldots n \text{ terms})}{\cos\alpha \,.\, \cos\beta \,.\, \cos\gamma \,..\, n \text{ factors}} = s_1 - s_3 + s_5 - s_7 + \text{etc}$$

where s_1 stands for the sum of $\tan\alpha + \tan\beta + \tan\gamma + \;.\; n$ terms etc, s_3 stands for the sum of the products of these tangents three at a time, and so on.

(11) Prove that $\tan(\alpha+\beta+\gamma+ \,.\,.\, n \text{ terms}) = \dfrac{s_1 - s_3 + s_5 - \text{etc.}}{1 - s_2 + s_4 - \text{etc.}}$

where $s_1, s_2, \ldots$ are defined in Ex. (10).

(12) Write down the last term of the numerator of the fraction in Ex. (11), (i) when n is even, (ii) when n is odd.

Use of De Moivre's Theorem.

13. It is known from the Theory of Equations that there are q different values of x, and no more, which satisfy the equation $x^q = a$, where a is real or of the form

$$A + \sqrt{(-1)}\, B.$$

We can prove that we may obtain q different values by De Moivre's theorem, and no more.

14. The expression $\cos\theta + \sqrt{-1}\sin\theta$ is unaltered if for θ we put $(\theta + 2r\pi)$, where r is an integer.

Hence $\qquad \cos\dfrac{\theta+2r\pi}{q} + \sqrt{-1}\sin\dfrac{\theta+2r\pi}{q}$,

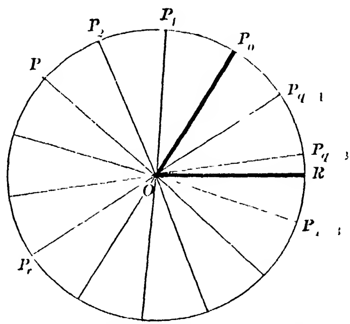
which is one of the values of

$$\{\cos(\theta+2r\pi) + \sqrt{-1}\sin(\theta+2r\pi)\}^{\frac{1}{q}},$$

is one of the values of

$$\{\cos\theta + \sqrt{-1}\sin\theta\}^{\frac{1}{q}}$$

15 *By giving to r the values* 0, 1, 2 . q − 1 *we obtain q different values of* $\cos\frac{\theta + 2r\pi}{q} + \sqrt{-1}\sin\frac{\theta + 2r\pi}{q}$; *and whatever integral value we give to r, we cannot obtain more than q different values.*



Take a circle, centre O and radius OR Let ROP_0 be the angle $\frac{\theta}{q}$. Divide the whole circumference of the circle starting from P_0, into q *equal arcs*, P_0P_1, P_1P_2, P_2P_3, etc.

Then each of the angles P_0OP_1, P_1OP_2, P_2OP_3, ...etc. is equal to $\frac{2\pi}{q}$; and in describing the angle $\left(\frac{\theta}{q} + \frac{2r\pi}{q}\right)$, the revolving line, starting from OR, turns first into the position OP_0 and then on through r of the angles P_0OP_1, P_1OP_2, etc. Hence, whatever integral value r may have, OP must stop in one of the q positions OP_0, OP_1, OP_2, etc. and it can stop in no other position.

Therefore the expression $\cos\frac{\theta + 2r\pi}{q} + \sqrt{-1}\sin\frac{\theta + 2r\pi}{q}$ *cannot have* **more than** q different values.

Also no two of these q positions are equi-sinal and at the same time equi-cosinal.

Therefore this expression has q *different* values.

Also by giving to r the values 0, 1, 2, ... $(q-1)$, in succession, OP will be made to stop in each of the q possible positions in turn.

Therefore by giving to r the values 0, 1, 2 . $(q-1)$ in succession, we obtain *the* q *different values* of the above expression. Q. E. D.

16. An expression of the form $A+\sqrt{-1}B$, where A and B are arithmetical quantities, can always be put into the form

$$r\{\cos a+\sqrt{-1}\sin a\}$$

Let $A=r\cos a$, $B=r\sin a$. Then

$A^2+B^2=r^2(\cos^2 a+\sin^2 a)=r^2$, and

$$\cos a=\frac{A}{r}=\frac{A}{+\sqrt{(A^2+B^2)}},\quad \sin a=\frac{B}{+\sqrt{(A^2+B^2)}};$$

whence a and r can always be found.

It will be convenient to take r positive; then we must take a in that quadrant which makes $\cos a$ of the same sign as A, and $\sin a$ of the same sign as B. [Cf. E. 148, 149.]

Example i. *Express* $1+\sqrt{-1}$ *in the form* $r(\cos a+\sqrt{-1}\sin a)$

Here $r\sin a=1$ and $r\cos a=1$, $\therefore r^2=2$, $\cos a=\sin a=\frac{1}{2}\sqrt{2}$

Example ii. *Express* $(-a)$ *in the form* $r(\cos a+i\sin a)$.

Here $r\cos a=-a$, $r\sin a=0$, $\therefore r^2=a^2$, $a=(2n+1)\pi$.

$\therefore -a=a\{\cos(2n+1)\pi+i\sin(2n+1)\pi\}$, where n is an integer.

Example iii. *Find all the values of* $16^{\frac{1}{4}}$.

$$16^{\frac{1}{4}}=2\times\sqrt[4]{1}$$

$$1=\cos 2n\pi+i\sin 2n\pi,\quad \sqrt[4]{1}=\cos\frac{n\pi}{2}+i\sin\frac{n\pi}{2}.$$

The four values of which are found by putting for n the values 0, 1, 2, 3 in succession.

They are $1, i, -1, -i$; hence the four values of $16^{\frac{1}{4}}$ are

$$\pm 2,\ \pm 2\sqrt{(-1)}.$$

Example IV. *Find all the values of* $\{\sqrt{3}-\sqrt{(-1)}\}^{\frac{1}{5}}$.

Let $\sqrt{3}-\sqrt{(-1)}=r(\cos\alpha+\iota\sin\alpha)$ then $r^2=4$ or $r=2$

$$\cos\alpha=\tfrac{1}{2}\sqrt{3},\quad \sin\alpha=-\tfrac{1}{2},\quad \therefore\ \alpha=-\tfrac{1}{6}\pi,$$

hence $$\{\sqrt{3}-\sqrt{(-1)}\}^{\frac{1}{5}}=\sqrt[5]{2}\left\{\cos\frac{2n\pi-\frac{1}{6}\pi}{5}+i\sin\frac{2n\pi-\frac{1}{6}\pi}{5}\right\}$$

and the five different values can be found by putting for n the values 0, 1, 2, 3, 4 in succession.

EXAMPLES. VII.

(1) Express (i) $1-\sqrt{-1}$,

(ii) $\sqrt{3}+\sqrt{-1}$,

(iii) $1+\sqrt{3}\sqrt{-1}$

each in the form $r(\cos\alpha+\sqrt{-1}\sin\alpha)$.

(2) Find all the values of

(i) $(4\sqrt{2}+4\sqrt{2}\sqrt{-1})^{\frac{1}{3}}$,

(ii) $(4\sqrt{3}+4\sqrt{-1})^{\frac{1}{3}}$,

(iii) $(\sqrt{3}+\sqrt{-1})^{\frac{1}{5}}$.

(3) Find all the values (i) of $1^{\frac{1}{4}}$,

(ii) of $32^{\frac{1}{5}}$,

(iii) of $27^{\frac{1}{3}}$.

17. If we express any arithmetical quantity a in the form of a De Moivre's expression we obtain

$$a(\cos 2r\pi+\sqrt{-1}\sin 2r\pi),$$

i.e. the product of a by the De Moivre's expression for unity. Therefore the n nth roots of any arithmetical quantity a are found by multiplying the arithmetical nth root of a by each of the n nth roots of unity in succession.

The nth roots of unity are therefore important.

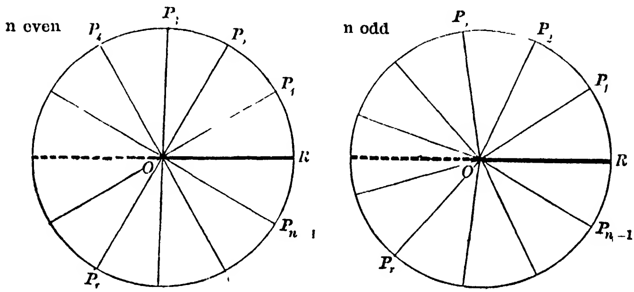
Find all the values of $\sqrt[n]{1}$, or *solve the equation* $x^n - 1 = 0$. In other words, *find the factors of* $x^n - 1$.

Since $$\cos 2r\pi + \sqrt{-1} \sin 2r\pi = 1.$$

It follows that $x^n = \cos 2r\pi + \sqrt{-1} \sin 2r\pi$, where r is an integer, and

therefore $$x = \cos \frac{2r\pi}{n} + \sqrt{-1} \sin \frac{2r\pi}{n}.$$

This result is best discussed by means of a figure.



I. Let n be a whole number.

Let the angle $ROP_1 = \frac{2\pi}{n}$. On the circumference of the circle centre O and radius OR, measure off arcs P_1P_2, P_2P_3, etc. each equal to RP_1. Then since $n \, . \, ROP_1 = 2\pi$, n of these arcs will occupy the whole circumference, and OP_n will coincide with OR. Also, if r be a whole number, in describing the angle $\frac{2r\pi}{n}$ the revolving line, starting from OR, must stop in one of the positions OP_1, OP_2, etc. and in no other. No two of these positions are both equi-sinal and equi-cosinal.

Thus the expression $$\cos \frac{2r\pi}{n} + \sqrt{-1} \sin \frac{2r\pi}{n}$$

has n different values, and no more; and these values can be found by giving r in succession the values $0, 1, 2 \ldots n-1$.

When $r = 0$, $x = 1$: when n is odd this is the only arithmetical value; when n is even, there are two arithmetical values; for let $n = 2m$, then when $r = m$, $x = -1$.

In any case, the angles ROP_1 and ROP_{n-1} are equi-cosinal, and $\sin ROP_1 = -\sin ROP_{n-1}$. The same thing is true of ROP_2, ROP_{n-2}, and of ROP_3, ROP_{n-3}, and so on.

Hence $x - \cos\frac{2\pi}{n} - \sqrt{-1}\sin\frac{2\pi}{n}$, and $x - \cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}$,

are factors of $x^n - 1$. Their product is

$$\left(x - \cos\frac{2\pi}{n}\right)^2 + \sin^2\frac{2\pi}{n}, \text{ i.e. } x^2 - 2x\cos\frac{2\pi}{n} + 1.$$

Hence we obtain that m being a whole number

$x^{2m} - 1 = (x^2 - 1)\left(x^2 - 2x\cos\frac{2\pi}{2m} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{2m} + 1\right)$ m quadratic factors,

$x^{2m+1} - 1 = (x - 1)\left(x^2 - 2x\cos\frac{2\pi}{2m+1} + 1\right)\left(x^2 - 2x\cos\frac{4\pi}{2m+1} + 1\right)$.

m quadratic factors.

[Note. Let $\alpha = \cos\frac{2\pi}{n} + \sqrt{-1}\sin\frac{2\pi}{n}$,

then $\cos\frac{2r\pi}{n} + \sqrt{-1}\sin\frac{2r\pi}{n} = \alpha^r$.

Therefore the roots of the equation $x^n - 1 = 0$ are $1, \alpha, \alpha^2, \alpha^3 \quad \alpha^{n-1}$.]

II. When n is a fraction in its lowest terms $=\frac{p}{q}$. Then $x^{\frac{p}{q}} - 1 = 0$, or $x^p - 1^q = 0$, or $x^p - 1 = 0$. This is the same as the case already discussed.

III. When n is incommensurable (e.g. $\sqrt{2}$). Then as before

$$x = \cos\frac{2r\pi}{n} + \sqrt{-1}\sin\frac{2r\pi}{n}.$$

In this case, r being an integer and n incommensurable, $\frac{2r\pi}{n}$ can never be an exact multiple of 2π. The angles will therefore not recur geometrically and the equation will have only one real root, viz. 1, and an unlimited number of symbolical roots.

EXAMPLES. VIII.

(1) Find the roots of the equation $x^5-1=0$.

(2) Find the quadratic factors of x^8-1

(3) Write down the quadratic factors of $x^{13}-1$

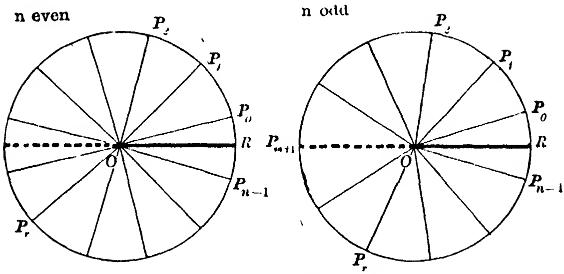
(4) Solve the equation $x^6-1=0$.

(5) Give the general quadratic factor of $x^{20}-a^{20}$.

(6) Find all the values of $\sqrt[12]{1}$.

To find the Quadratic factors of $x^n+1=0$.

Here $$x=\cos\frac{\pi+2r\pi}{n}+i\sin\frac{\pi+2r\pi}{n}.$$



In the figure $ROP_0=\frac{\pi}{n}$, $P_0OP_1=\frac{2\pi}{n}$, and n angles each equal to P_0OP_1 make up 2π; OR bisects P_0OP_{n-1}. Also ROP_0 and ROP_{n-1} are equi-cosinal, while $\sin ROP_0=-\sin ROP_{n-1}$, the same relation holds good for any two angles equi-distant from OR.

$\therefore \left(x-\cos\frac{\pi+2r\pi}{n}-i\sin\frac{\pi+2r\pi}{n}\right)$, and $\left(x-\cos\frac{\pi+2r\pi}{n}+i\sin\frac{\pi+2r\pi}{n}\right)$.
are factors of x^n+1.

Therefore their product viz. $\left(x^2-2x\cos\frac{\pi+2r\pi}{n}+1\right)$
is the form of the general quadratic factor of x^n+1.

When n is even and $=2m$ there are m such factors.

When n is odd and $=2m+1$ there are m such factors; the remaining factor is $x+1$, as is clear from the figure.

EXAMPLES. IX.

(1) Find the roots of the equation $x^4+1=0$, and write down the quadratic factors of x^4+1.

(2) Write down the quadratic factors of x^6+1.

(3) Write down the general quadratic factor of $x^{20}+1=0$.

(4) Find all the values of $\sqrt[6]{-1}$. (5) Find the factors of $x^{13}+1$.

(6) Find a general expression for all the values of $\sqrt[n]{-1}$

*MISCELLANEOUS EXAMPLES. X.

(1) Prove that

$$\sin(a_1+a_2+a_3 \ldots n \text{ terms}) = s_1c_{n-1} - s_3c_{n-3} + s_5c_{n-5} - \text{etc}$$

where s_rc_{n-r} stands for the sum of the products of the sines taken r together each multiplied by the product of the remaining $n-r$ cosines.

(2) With the notation of Ex. 1, prove that

$$\cos(a_1+a_2+a_3 \ldots n \text{ terms}) = c_n - c_{n-2}s_2 + c_{n-4}s_4 - \text{etc.}$$

(3) Write down the expansion of

$$\sin(\alpha+\beta+\gamma+\delta+\epsilon) \text{ and of } \cos(\alpha+\beta+\gamma+\delta+\epsilon)$$

(4) Prove that in the series of expressions formed by giving to r in $\left(\cos\dfrac{\theta+2r\pi}{q}+i\sin\dfrac{\theta+2r\pi}{q}\right)$ the values $0, 1, 2, 3 \ldots (q-1)$ in succession, the product of any two equidistant from the beginning and the end is constant.

(5) One value of $(\sqrt{3}+\sqrt{-1})^{\frac{15}{2}}$ is $-2^7(\sqrt{-1}+1)$.

(6) From the identity

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1,$$

deduce by assuming $x=\cos 2\theta+i\sin 2\theta$, and corresponding assumptions for a, b and c that $\dfrac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\sin 2(\theta-\alpha)+$ two similar expressions$=0$.

(7) Prove that the n nth roots of unity form a series in G.P.

CHAPTER III

RESULTS OF DE MOIVRE'S THEOREM.

18. We proceed to deduce some important results from De Moivre's Theorem.

We shall generally in this chapter write i for $\sqrt{-1}$.

19. By Art. 12, when n is an integer we have

$$\cos n\theta + i \sin n\theta = (\cos\theta + i\sin\theta)^n.$$

Expand the right-hand side of this identity by the binomial theorem, remembering that $i^2 = -1$ and that $i^4 = +1$. Equate the real part of the result to $\cos n\theta$. This gives us

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{\lfloor 2}\cos^{n-2}\theta \,.\, \sin^2\theta$$
$$+ \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\cos^{n-4}\theta \,.\, \sin^4\theta - \text{etc.}$$

Equate the imaginary part to $i \sin n\theta$. This gives us

$$\sin n\theta = n\cos^{n-1}\theta \,.\, \sin\theta - \frac{n(n-1)(n-2)}{\lfloor 3}\cos^{n-3}\theta \,.\, \sin^3\theta$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}\cos^{n-5}\theta \,.\, \sin^5\theta - \text{etc.}$$

20. In the above n is a positive integer, and the last terms in the series for $\cos n\theta$ and for $\sin n\theta$ will be different according as n is even or odd.

The last term in the expansion of $\{\cos\theta + i\sin\theta\}^n$ is $i^n \sin^n\theta$, which is real when n is even and imaginary when n is odd. The last term but one is $i^{n-1} n\cos\theta\sin^{n-1}\theta$, which is real when n is odd and imaginary when n is even.

Thus the last term in the expansion of $\sin n\theta$ is

$$i^{n-1}\sin^n\theta \text{ or } (-1)^{\frac{1}{2}(n-1)}\sin^n\theta \text{ when } n \text{ is } odd,$$

and $\quad i^{n-2} n\cos\theta\sin^{n-1}\theta$ or $(-1)^{\frac{1}{2}(n-2)} n\cos\theta\sin^{n-1}\theta$ when n is *even*.

EXAMPLES. XI.

Prove the following statements:

(1) $\sin 4\theta = 4\cos^3\theta \,.\, \sin\theta - 4\cos\theta \,.\, \sin^3\theta$.

(2) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta \,.\, \sin^2\theta + \sin^4\theta$.

(3) $\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$.

(4) $\sin 9\theta = 9\cos^8\theta\sin\theta - 84\cos^6\theta\sin^3\theta + 126\cos^4\theta\sin^5\theta$
$- 36\cos^2\theta\sin^7\theta + \sin^9\theta$.

(5) The last term in the expansion of $\cos 10\theta$ is $-\sin^{10}\theta$.

(6) The last term in the expansion of $\sin 12\theta$ is

$$-12\cos\theta \,.\, \sin^{11}\theta.$$

(7) When n is even the last term in the expansion of $\cos n\theta$ is

$$(-1)^{\frac{n}{2}}\sin^n\theta.$$

(8) When n is odd the last term in the expansion of $\cos n\theta$ is

$$(-1)^{\frac{n-1}{2}} n\cos\theta \,.\, \sin^{n-1}\theta.$$

(9) $$\tan n\theta = \frac{n\tan\theta - \frac{n(n-1)(n-2)}{\lfloor 3}\tan^3\theta + }{1 - \frac{n(n-1)}{\lfloor 2}\tan^2\theta + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\tan^4\theta - }.$$

(10) $$\cot n\theta = \frac{\cot^n\theta - \frac{n(n-1)}{\lfloor 2}\cot^{n-2}\theta + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\cot^{n-4}\theta}{n\cot^{n-1}\theta - \frac{n(n-1)(n-2)}{\lfloor 3}\cot^{n-3}\theta + \ldots}.$$

Exponential Values of Sine and Cosine.

21. By De Moivre's Theorem, when n is any commensurable number, and x any angle,

$$(\cos nx + i \sin nx) \text{ is a value of } (\cos x + i \sin x)^n.$$

For x put the unit of angular measurement; then

$$(\cos n + i \sin n) \text{ is a value of } (\cos 1 + i \sin 1)^n.$$

Let k stand for $(\cos 1 + i \sin 1)$, then

$$(\cos n + i \sin n) \text{ is a value of } k^n,$$

where k is independent of n.

Whatever other values $(\cos 1 + i \sin 1)^n$ may have, in what follows we shall only use the value $(\cos n + i \sin n)$.

22. This important result is a symbolical statement of the fact that **expressions of the form** $\cos n + i \sin n$ **are combined by the laws of indices.**

23. Let the unit of angle be a radian [E. 59.]

Then since $\cos\theta + i \sin\theta = k^\theta$, [Art. 21.]

and consequently $\cos\theta - i \sin\theta = k^{-\theta}$,

where k is independent of θ,

$$\therefore\ 2i \sin\theta = k^\theta - k^{-\theta},$$

$$= 2\left\{\theta \log_e k + \frac{1}{\underline{|3}}\,\theta^3 (\log_e k)^3 + \text{etc.}\right\}. \qquad \text{[Art. 3.]}$$

Hence
$$i\,\frac{\sin\theta}{\theta} = \log_e k + \frac{1}{\underline{|3}}\,\theta^2 (\log_e k)^3 + \text{etc.}$$

$$= \log_e k + \theta^2 . R;$$

where R is finite for all values of θ (since $\sin\theta$ is always less than θ, and $\therefore$ $\log_e k$ is finite)

Let θ be infinitely diminished. Then, since θ is the circular measure of the angle, the limit of $\frac{\sin\theta}{\theta}$ is 1. [E. 290.]

Also the limit of the right-hand side is $\log_e k$.

Hence $$i = \log_e k,$$

or, $$k = e^{\sqrt{-1}}.$$

Therefore, when θ is the circular measure of the angle,

$$\cos\theta + \sqrt{-1}\sin\theta = e^{\sqrt{-1}\theta}.$$

24. Since $\cos\theta + i\sin\theta = e^{i\theta}$ and $\cos\theta - i\sin\theta = e^{-i\theta}$,

$$\therefore \quad 2\cos\theta = e^{i\theta} + e^{-i\theta}; \text{ and } 2i\sin\theta = e^{i\theta} - e^{-i\theta}.$$

Hence $$\frac{e^{i\theta} + e^{-i\theta}}{2} \text{ and } \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

are **exponential values** of the cosine and sine respectively, *when the angle is expressed in circular measure.*

These results may be applied to prove any *general* formula in Elementary Trigonometry.

Example. Prove $\dfrac{\sin 2a}{1+\cos 2a} = \tan a$

We have $$\frac{2i\sin 2a}{2+2\cos 2a} = \frac{e^{2ia} - e^{-2ia}}{2+e^{2ia}+e^{-2ia}} = \frac{(e^{ia}+e^{-ia})(e^{ia}-e^{-ia})}{(e^{ia}+e^{-ia})^2}$$

$$= \frac{e^{ia}-e^{-ia}}{e^{ia}+e^{-ia}} = \frac{2i\sin a}{2\cos a} = i\tan a. \quad \text{Q.E.D.}$$

EXAMPLES. XII.

Use the exponential values of the sine and cosine to prove the following: (1) $\cos^2 a + \sin^2 a = 1$. (2) $\cos 2a = \cos^2 a - \sin^2 a$.

(3) $\sin\theta = -\sin(-\theta)$. (4) $\cos\theta = \cos(-\theta)$.

(5) $\cos(a+\beta) \cdot \cos(a-\beta) = \cos^2 a - \sin^2\beta = \cos^2\beta - \sin^2 a$.

(6) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. (7) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

(8) $2\cos na \cdot \cos a = \cos(n+1)a + \cos(n-1)a$.

(9) $2\sin na(-1)2^2\sin^2 a = 2\sin(n+2)a - 4\sin na + 2\sin(n-2)a$.

(10) $2\cos na(-1)2^2\sin^2 a = 2\cos(n+2)a - 4\cos na + 2\cos(n-2)a$.

25. The results

$$2\cos\theta = e^{i\theta} + e^{-i\theta}, \quad 2i\sin\theta = e^{i\theta} - e^{-i\theta}$$

may be used to simplify expressions containing $\sqrt{-1}$.

Example 1. *Reduce* $\cos(a + i\beta)$ *to the form* $A + iB$.

$$2\cos(a+i\beta) = e^{ia-\beta} + e^{-ia+\beta} = e^{-\beta}\,.\,e^{ia} + e^{\beta}\,.\,e^{-ia}$$

$$= e^{-\beta}(\cos a + i\sin a) + e^{\beta}(\cos a - i\sin a)$$

$$= \cos a\,(e^{\beta} + e^{-\beta}) - i\sin a\,(e^{\beta} - e^{-\beta}).$$

This is in the required form.

Example 2. *Express* $\log(a + ib)$ *in the form* $A + iB$.

Let $$a + ib = r(\cos a + i\sin a).$$

Then (Art. 16), $$r^2 = a^2 + b^2,\ \tan a = \frac{b}{a}.$$

Thus, $$\log(a+ib) = \log\{r(\cos a + i\sin a)\}$$

$$= \log r + \log e^{ia}$$

$$= \log r + ia$$

$$= \tfrac{1}{2}\log(a^2 + b^2) + i\tan^{-1}\frac{b}{a}.$$

This is in the required form.

Example 3. *Reduce* $(a + ib)^{a+i\beta}$ *to the form* $A + iB$.

Let $$a + ib = r(\cos\gamma + i\sin\gamma).$$

Then $$r^2 = a^2 + b^2,\ \tan\gamma = \frac{b}{a}.$$

And $$(a+ib)^{a+i\beta} = r^{a+i\beta}\,.\,e^{i\gamma(a+i\beta)}$$

$$= r^{a}\,.\,r^{i\beta}\,.\,e^{i\gamma a}\,.\,e^{-\beta\gamma}$$

$$= r^{a}\,.\,e^{-\beta\gamma}\,.\,e^{i\beta\log r}\,.\,e^{i\gamma a} \quad [\text{For } r = e^{\log r}]$$

$$= r^{a}e^{-\beta\gamma}\,.\,e^{i(\beta\log r + a\gamma)}$$

$$= r^{a}e^{-\beta\gamma}\{\cos(\beta\log r + a\gamma) + i\sin(\beta\log r + a\gamma)\}.$$

This is in the required form.

EXAMPLES. XIII.

Prove the following statements:

(1) $\cos(\alpha+i\beta)+i\sin(\alpha+i\beta)=e^{-\beta}(\cos\alpha+i\sin\alpha)$.

(2) $2\sin(\alpha+i\beta)=(e^{\beta}+e^{-\beta})\sin\alpha+i(e^{\beta}-e^{-\beta})\cos\alpha$.

(3) $\cos(\alpha+i\beta)-i\sin(\alpha+i\beta)=e^{\beta}(\cos\alpha-i\sin\alpha)$.

(4) $4\cos(\alpha+i\beta)\,.\,\cos(\alpha-i\beta)=e^{2\beta}+e^{-2\beta}+2\cos 2\alpha$.

(5) $4\sin(\alpha+i\beta)\,.\,\cos(\alpha-i\beta)=2\sin 2\alpha+i(e^{2\beta}-e^{-2\beta})$.

(6) $\sec(\alpha+i\beta)=\dfrac{2(e^{\beta}+e^{-\beta})\cos\alpha+2i(e^{\beta}-e^{-\beta})\sin\alpha}{e^{2\beta}+2\cos 2\alpha+e^{-2\beta}}$.

(7) $\tan(\alpha+i\beta)=\dfrac{2\sin 2\alpha+i(e^{2\beta}-e^{-2\beta})}{e^{2\beta}+2\cos 2\alpha+e^{-2\beta}}$.

(8) $(\sqrt{-1})^{\sqrt{-1}}=e^{-\frac{\pi}{2}}$.

(9) Express a^{p+iq} in the form $A+iB$.

(10) Express $(a+ib)^{i}$ in the form $A+iB$.

(11) $\log\dfrac{a+ib}{a-ib}=2i\tan^{-1}\dfrac{b}{a}$.

(12) $\log\dfrac{\sin(x+iy)}{\sin(x-iy)}=2i\tan^{-1}\left\{\cot x\,\dfrac{e^{y}-e^{-y}}{e^{y}+e^{-y}}\right\}$.

(13) $\log\dfrac{\cos(x-iy)}{\cos(x+iy)}=2i\tan^{-1}\left\{\tan x\,\dfrac{e^{y}-e^{-y}}{e^{y}+e^{-y}}\right\}$.

(14) $\log 4\sin^2(x+iy)$

$$=\log(e^{2y}-2\cos 2x+e^{-2y})+2i\tan^{-1}\left\{\cot x\,\frac{e^{y}-e^{-y}}{e^{y}+e^{-y}}\right\}.$$

(15) $\log 4\cos^2(\alpha+i\beta)$

$$=\log(e^{2\beta}+2\cos 2\alpha+e^{-2\beta})-2i\tan^{-1}\left\{\tan\alpha\,\frac{e^{\beta}-e^{-\beta}}{e^{\beta}+e^{-\beta}}\right\}.$$

(16) $a^{\sqrt{i}}=a^{\cos 45^{\circ}}\{\cos(\sin 45^{\circ}\log a)+i\sin(\sin 45^{\circ}\log a)\}$.

(17) Express $a^{\sqrt[3]{i}}$ in the form $A+iB$.

(18) Express $(a+ib+c^{i})^{\alpha+i\beta}$ in the form $A+iB$.

26. Since

$$\cos\theta + i\sin\theta = e^{i\theta} = 1 + i\theta - \frac{\theta^2}{\underline{|2}} - \frac{i\theta^3}{\underline{|3}} + \frac{\theta^4}{\underline{|4}} + \text{etc.},$$

we obtain by equating the real and imaginary parts

$$\cos\theta = 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \text{etc.},$$

$$\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \text{etc.}$$

These results are very important.

In the next chapter will be found a proof independent of $\sqrt{-1}$ and a collection of examples.

$e^{i\theta}$ is a Periodic Function.

27. $\cos\theta$ and $\sin\theta$, and therefore also $\cos\theta + i\sin\theta$, repeat their values every time θ is increased by 2π. Therefore $e^{i\theta}$ also repeats its values every time θ is increased by 2π.

Also, as θ passes from 0 to 2π no two values of $\cos\theta + i\sin\theta$ are the same.

When a function of θ repeats every possible value in exactly the same order each time θ is increased by a certain value λ, it is said to be **periodic**, and λ is called its **period.**

If we are given a particular value of such a function of θ, we can find an unlimited number of values for θ (each of the form $a + n\lambda$, where n is a whole number,) for each of which this function will have that given value.

Also, in the particular case we are considering, as θ changes from 0 to 2π, none of the values of $e^{i\theta}$ are repeated. In other words, there are no two values of θ *in the same period* for which $e^{i\theta}$ has equal values.

Example. Given that $\tan\theta = a$, and that α is one angle whose tangent is a, then we know that $\theta = \alpha + n\pi$, where n is a whole number.

Hence $\tan\theta$ is a periodic function of θ whose period is π.

28. Hence, if $e^{\iota\theta} = e^{\iota\alpha}$, we know that θ and α differ by some multiple of 2π, *i.e.* that $\theta = \alpha + 2n\pi$, where n is a whole number, and the value of n cannot be decided without some further datum.

Example. Since $\quad 2\iota \sin\theta = e^{\iota\theta} - e^{-\iota\theta}$, and $\sin\pi = 0$,

therefore $\quad e^{\iota\pi} - e^{-\iota\pi} = 0$ or $e^{\iota\pi} = e^{-\iota\pi}$.

This means that π and $-\pi$ are two values of θ for which the periodic function $e^{\iota\theta}$ has the same value. And since the period is 2π, π and $-\pi$ should differ by $2n\pi$. In this case n is clearly 1.

29. The same thing may be stated thus:

since $\quad \cos\alpha + i\sin\alpha = \cos(\alpha + 2r\pi) + i\sin(\alpha + 2r\pi)$,

$$\therefore\ e^{\iota\alpha} = e^{\iota\alpha + 2\iota r\pi} = e^{\iota\alpha} \cdot e^{2\iota r\pi}.$$

Therefore $e^{2\iota r\pi} = 1$ (as is also evident since

$$\cos 2r\pi + i\sin 2r\pi = 1).$$

Hence **unity has** one real logarithm, viz. 0, and also **an unlimited number of symbolical logarithms** each equal to $2ir\pi$, where r is some integer.

30. Again, $\quad a = a \times 1 = a \times e^{2\iota r\pi} = e^{\log_e a + 2\iota r\pi}$.

Hence **every real positive quantity** a **has** one real logarithm, and also **an unlimited number of symbolical logarithms,** which differ by $2ir\pi$, where r is an integer.

These symbolical logarithms do not interfere in any way with the theory of *arithmetical* logs explained in Chapter I.

Example. Prove that the equation $\sin\theta = 0$ *has no symbolical roots.*

Suppose that $\sin(\alpha + \sqrt{-1}\beta) = 0$ then

$$e^{i\alpha - \beta} - e^{-\iota\alpha + \beta} = 0, \text{ or } e^{2\iota\alpha - 2\beta} = 1 = e^{2\iota r\pi}.$$

$\therefore\ 2i\alpha - 2\beta = 2ir\pi$; $\qquad \therefore\ \beta = 0$; which proves the proposition.

EXAMPLES. XIV.

(1) Point out the fallacy in the following.

Since $2i \sin 3\pi = e^{i3\pi} - e^{-i3\pi}$ and $\sin 3\pi = 0$,

therefore $e^{i3\pi} = e^{-i3\pi}$; that is, $e^{6i\pi} = 1$; therefore $6\pi = 0$.

(2) Expose the fallacy in the following:

Let α be any angle, then since

$$\cos(\alpha - \pi) + i \sin(\alpha - \pi) = \cos(\alpha + \pi) + i \sin(\alpha + \pi),$$

$$\text{therefore } e^{i(\alpha - \pi)} = e^{i(\alpha + \pi)},$$

$$\text{hence } \alpha - \pi = \alpha + \pi; \text{ that is, } \pi = 0.$$

(3) Prove that the equation $\cos\theta = 0$ has no symbolical roots.

EXPANSION OF $\tan^{-1}x$ IN TERMS OF x

31. Since $2\cos\alpha = e^{i\alpha} + e^{-i\alpha}$,

and $2i\sin\alpha = e^{i\alpha} - e^{-i\alpha}$,

therefore $$i\tan\alpha = \frac{2i\sin\alpha}{2\cos\alpha} = \frac{e^{i\alpha} - e^{-i\alpha}}{e^{i\alpha} + e^{-i\alpha}}, \qquad \text{[Art. 24.]}$$

$$\therefore \frac{1 + i\tan\alpha}{1 - i\tan\alpha} = \frac{2e^{i\alpha}}{2e^{-i\alpha}} = e^{2i\alpha};$$

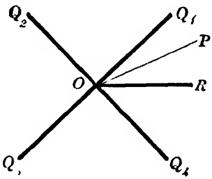
$$\therefore \log\frac{1 + i\tan\alpha}{1 - i\tan\alpha} = \log e^{2i\alpha} = 2i\alpha + 2in\pi. \qquad \text{[Art. 28.]}$$

Hence expanding the left-hand side by Art. 5, we have

$$2i\alpha + 2in\pi = 2\left\{i\tan\alpha + \tfrac{1}{3}(i\tan\alpha)^3 + \tfrac{1}{5}(i\tan\alpha)^5 + \text{etc.}\right\}$$

or $$\alpha + n\pi = \tan\alpha - \tfrac{1}{3}\tan^3\alpha + \tfrac{1}{5}\tan^5\alpha - \text{etc.}$$

This series is convergent if $\tan\alpha$ is equal to or less than unity. It is therefore arithmetically intelligible and true provided α lies between $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, or between $\frac{3\pi}{4}$ and $\frac{5\pi}{4}$,



and so on [*i.e.* provided OP stops within the right angle Q_4OQ_1 or within the right angle Q_2OQ_3 in the figure]

32. When α lies between $-\frac{1}{4}\pi$ and $\frac{1}{4}\pi$, the value of n is 0 For n is an *integer* (or zero), while the value of π is known to be greater than 3 [E. 37], and as α varies from $-\frac{1}{4}\pi$ to $\frac{1}{4}\pi$ the numerical value of the series

$$\tan\alpha - (\tfrac{1}{3}\tan^3\alpha - \tfrac{1}{5}\tan^5\alpha) - (\tfrac{1}{7}\tan^7\alpha - \tan^9\alpha) - \dots$$

is always less than $\tan\alpha$, and therefore less than 1.

Hence, when α lies between $-\frac{1}{4}\pi$ and $\frac{1}{4}\pi$, we have

$$\tan\alpha - (\tfrac{1}{3}\tan^3\alpha - \tfrac{1}{5}\tan^5\alpha) - (\tfrac{1}{7}\tan^7\alpha - \tfrac{1}{9}\tan^9\alpha) - \dots$$

This result is called **Gregory's Series.**

Similarly we can prove that when α lies between $\frac{3}{4}\pi$ and $\frac{5}{4}\pi$, $\quad \alpha - \pi = \tan\alpha - \frac{1}{3}\tan^3\alpha + \frac{1}{5}\tan^5\alpha$
and so on.

33. In this result put $\alpha = \frac{\pi}{4}$, and we obtain

$$\frac{\pi}{4} = 1 - \tfrac{1}{3} + \tfrac{1}{5} - \tfrac{1}{7} + \text{etc.}$$

34. The series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} +$ etc. is very slowly convergent; we shall therefore show how series which are more rapidly convergent may be obtained from Gregory's Series.

35. **Euler's Series.**

Since $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \tan^{-1}\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}} = \tan^{-1} 1 = \dfrac{\pi}{4}.$

Let $\alpha = \tan^{-1}\frac{1}{2}$, or $\tan\alpha = \frac{1}{2}$,

and
$$\alpha = \tan\alpha - \tfrac{1}{3}\tan^3\alpha + \tfrac{1}{5}\tan^5\alpha - \text{etc}$$
$$\therefore\ \alpha = \tfrac{1}{2} - \tfrac{1}{3}\cdot\frac{1}{2^3} + \tfrac{1}{5}\cdot\frac{1}{2^5} - \text{etc.}$$

Let $\beta = \tan^{-1}\frac{1}{3}$, then
$$\beta = \tfrac{1}{3} - \tfrac{1}{3}\cdot\frac{1}{3^3} + \tfrac{1}{5}\cdot\frac{1}{3^5} - \text{etc.} \qquad \text{[Art. 31.]}$$

and $\frac{1}{4}\pi = \alpha + \beta =$ the sum of these two series

36. **Machin's Series.**

Since $2\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{5}{12}$ and $2\tan^{-1}\frac{5}{12} = \tan^{-1}\frac{120}{119}$,
$$\therefore\ 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{239} = \tfrac{1}{4}\pi.$$

Hence
$$\frac{\pi}{4} = 4\left\{\tfrac{1}{5} - \tfrac{1}{3}\cdot\frac{1}{5^3} + \tfrac{1}{5}\cdot\frac{1}{5^5} - \text{etc.}\right\}$$
$$-\left\{\tfrac{1}{239} - \tfrac{1}{3}\cdot\frac{1}{239^3} + \tfrac{1}{5}\cdot\frac{1}{239^5} - \text{etc.}\right\}. \qquad \text{[Art. 31.]}$$

NOTE. It may be shewn that
$$\tan^{-1}\tfrac{1}{239} = \tan^{-1}\tfrac{1}{70} - \tan^{-1}\tfrac{1}{99},$$

so that
$$\frac{\pi}{4} = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{70} + \tan^{-1}\tfrac{1}{99}.$$

The labour of calculating the numerical value of the series for $\tan^{-1}\frac{1}{70}$ and for $\tan^{-1}\frac{1}{99}$ is less than that of calculating the numerical value of the series for $\tan^{-1}\frac{1}{239}$.

Example. *Find the numerical value of π to 6 figures* by Machin's series. We have

$$\frac{1}{4}\pi = 4\left\{\frac{1}{5} - \frac{1}{3}\left(\frac{1}{5}\right)^3 + \frac{1}{5}\left(\frac{1}{5}\right)^5 - \frac{1}{7}\left(\frac{1}{5}\right)^7 + \quad\right\} - \frac{1}{239} + \frac{1}{3}\left(\frac{1}{239}\right)^3 -$$

$$= 4\{\cdot 2 - \cdot 0026666\ldots + \cdot 000064 - \cdot 0000018 \quad\} - \cdot 0041841.$$

$$= 4\{\cdot 1973956\ldots\} - \cdot 0041841$$

$$= \cdot 7853983.$$

Therefore $\pi = 4 \times \cdot 7853983 \quad = \underline{3\cdot 14159}32$

EXAMPLES. XV.

(1) Prove that $\dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}\left\{1 - \dfrac{1}{3}\cdot\dfrac{1}{3} + \dfrac{1}{5}\cdot\dfrac{1}{3^2} - \dfrac{1}{7}\cdot\dfrac{1}{3^3} + \text{etc.}\right\}$

(2) Prove that $4\tan^{-1}\frac{1}{5} = \tan^{-1}\frac{120}{119}$.

Hence prove that if $\frac{1}{4}\pi = 4\tan^{-1}\frac{1}{5} - x$,

then $x = \tan^{-1}\frac{1}{239}$.

(3) Prove that

$$\frac{\pi}{8} = \frac{1}{1\,.\,3} + \frac{1}{5\,.\,7} + \frac{1}{9\,.\,11} + \text{etc.},$$

hence calculate the value of π to 2 decimal places.

(4) Calculate the value of π to 3 decimal places by the aid of Euler's series.

(5) Calculate the value of π to 3 decimal places by the aid of Machin's series.

(6) Prove that $\frac{1}{4}\pi = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$

(7) *Defining* $\tanh a$ as $\dfrac{e^a - e^{-a}}{e^a + e^{-a}}$,

prove that

$$a = \tanh a + \tfrac{1}{3}\tanh^3 a + \tfrac{1}{5}\tanh^5 a + \text{etc.}\ldots$$

(8) Prove that when α lies between $\frac{7}{4}\pi$ and $\frac{9}{4}\pi$

$$\alpha - 2\pi = \tan\alpha - \tfrac{1}{3}\tan^3\alpha + \tfrac{1}{5}\tan^5\alpha$$

Expansion of $(2\cos\theta)^n$ and of $(2\sin\theta)^n$.

37. Let $$\cos\theta + i\sin\theta = x,$$

then $$\cos\theta - i\sin\theta = \frac{1}{x} = x^{-1}, \qquad \text{[Art. 12]}$$

and $$2\cos\theta = x + x^{-1},\quad 2i\sin\theta = x - x^{-1}.$$

[It should be observed that in the equation $2\cos\theta = x + x^{-1}$, either x or θ must be symbolical. For if θ be real $2\cos\theta$ is less than 2. If x be real $x + x^{-1}$ is numerically greater than 2. x of course stands for $e^{i\theta}$.]

Also, $$x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta, \quad \text{[Art. 12.]}$$
$$x^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta.$$
$$\therefore\ 2\cos n\theta = x^n + x^{-n},\ \text{and}\ 2i\sin n\theta = x^n - x^{-n}$$

Hence $(2\cos\theta)^n = (x + x^{-1})^n$

$$= x^n + n\,.\,x^{n-2} + \frac{n(n-1)}{\underline{|2}}x^{n-4} + \text{etc.} + nx^{-(n-2)} + x^{-n}$$

$$= (x^n + x^{-n}) + n(x^{n-2} + x^{-(n-2)}) + \frac{n(n-1)}{\underline{|2}}(x^{n-4} + x^{-(n-4)}) + \text{etc.}$$

$$= 2\cos n\theta + n\,.\,2\cos(n-2)\theta + \frac{n(n-1)}{\underline{|2}}2\cos(n-4)\theta + \text{etc.}$$

Also $$(2i\sin\theta)^n = (x - x^{-1})^n.$$

First let n be **even.** Then the expansion of $(x - x^{-1})^n$ is

$$x^n + x^{-n} - n(x^{n-2} + x^{-(n-2)}) + \frac{n(n-1)}{\underline{|2}}(x^{n-4} + x^{-(n-4)}) - \text{etc.}$$

$$\therefore\ (2i\sin\theta)^n = 2\cos n\theta - n2\cos(n-2)\theta + \frac{n(n-1)}{\underline{|2}}2\cos(n-4)\theta$$
$$- \text{etc.}$$

Next let n be **odd.** Then the expansion of $(x - x^{-1})^n$ is

$$x^n - x^{-n} - n(x^{n-2} - x^{-(n-2)}) + \frac{n(n-1)}{\underline{|2}}(x^{n-4} - x^{-(n-4)}) - \text{etc.}$$

$$\therefore\ (2i\sin\theta)^n = 2i\ \sin n\theta - n\,.\,2i\sin(n-2)\,\theta + \frac{n(n-1)}{\lfloor 2}\,2i\sin(n-4)\,\theta - \text{etc.}$$

Whence dividing by i and putting $i^2 = -1$ we have

$$(-1)^{\frac{n-1}{2}}2^n\sin^n\theta = 2\sin n\theta - n\,.\,2\sin(n-2)\,\theta + \text{etc.}$$

It must be noticed that when the last term is independent of θ, the factor 2 is omitted.

NOTE. When n is **even** the last term is independent of θ for then the expansion of $(a+x)^n$ [which has $n+1$ terms] has a *middle* term. Hence the last term in either expansion is independent of θ when n is even and is

$$\frac{n(n-1)(n-2)\ldots(\frac{1}{2}n+1)}{\lfloor\frac{1}{2}n} \quad\text{or}\quad (-1)^{\frac{n}{2}}\frac{n(n-1)(n-2)\ldots(\frac{1}{2}n+1)}{\lfloor\frac{1}{2}n}.$$

When n is **odd** the last term in the expansion of $\cos^n\theta$ is

$$+\frac{n(n-1)\ldots\frac{1}{2}(n+3)}{\lfloor\frac{1}{2}(n-1)}\,2\cos\theta,$$

and the last term in the expansion of $\sin\theta$ is

$$(-1)^{\frac{n-1}{2}}\frac{n(n-1)\ldots\frac{1}{2}(n+3)}{\lfloor\frac{1}{2}n-1}\,2\sin\theta.$$

EXAMPLES. XVI.

Prove that

(1) $128\cos^8\theta = \cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35.$

(2) $64\cos^7\theta = \cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta.$

(3) $-64\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta.$

(4) $-512\sin^{10}\theta = \cos 10\theta - 10\cos 8\theta + 45\cos 6\theta - 120\cos 4\theta + 210\cos 2\theta - 126.$

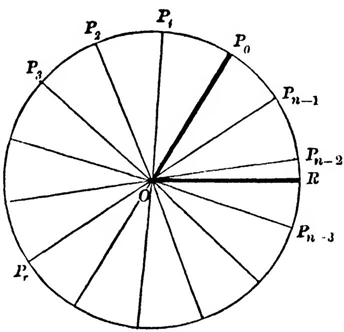
RESOLUTION INTO FACTORS OF $x^{2n} - 2x^n \cos n\theta + 1$.

38. *To resolve* $x^{2n} - 2x^n \cos n\theta + 1$ *into factors, when* n *is a whole number.*

Since $x^{2n} - 2x^n \cos n\theta + 1 = (x^n - \cos n\theta)^2 + \sin^2 n\theta$

$$= \{(x^n - \cos n\theta) + i \sin n\theta\}\{(x^n - \cos n\theta) - i \sin n\theta\}$$
$$= \{x^n - (\cos n\theta - i \sin n\theta)\}\{x^n - (\cos n\theta + i \sin n\theta)\}$$
$$= \{x^n - (\cos\theta - i\sin\theta)^n\}\{x^n - (\cos\theta + i\sin\theta)^n\},$$

therefore $x - (\cos\theta - i\sin\theta)$ and $x - (\cos\theta + i\sin\theta)$ are factors of $x^{2n} - 2x^n \cos n\theta + 1$.



Therefore also their product, *i.e.* $x^2 - 2x\cos\theta + 1$, is a factor of $x^{2n} - 2x^n \cos n\theta + 1$.

And since $\cos n\theta$ is unaltered if for θ we write $\theta + \dfrac{2r\pi}{n}$ it follows that $x^2 - 2x\cos\left(\theta + \dfrac{2r\pi}{n}\right) + 1$ is also a factor.

In the above figure let $ROP_0 = \theta$, and let the whole circumference, starting from P_0, be divided into n equal arcs P_0P_1, $P_1P_2 \dots P_{n-1}P_0$; so that, whatever be the integral value of r, the angle $\theta + \frac{2r\pi}{n}$ is represented by one of the angles ROP_0, ROP_1, etc.

Hence *in general* there are n different values and no more of $\cos\left(\theta + \frac{2r\pi}{n}\right)$.

[The exceptions are (i) when one of the points P_r coincides with R, (ii) when R *bisects* one of the arcs P_rP_{r-1}; *i e.* (i) when $n\theta = 2r\pi$, (ii) when $n\theta = (2r+1)\pi$, and in these cases $x^{2n} - 2x^n \cos n\theta + 1$ reduces (i) to the form $(x^n - 1)^2$ or (ii) to the form $(x^n + 1)^2$, the factors of these forms have been discussed on pp. 20, 21.]

And the n different values are found by giving to r the values 0, 1, 2 ... $(n-1)$ in succession.

Hence the n quadratic factors of $x^{2n} - 2x^n \cos n\theta + 1$ are

$$(x^2 - 2x\cos\theta + 1)\left\{x^2 - 2x\cos\left(\theta + \frac{2\pi}{n}\right) + 1\right\} \times$$
$$\times \left\{x^2 - 2x\cos\left(\theta + \frac{n-1}{n}2\pi\right) + 1\right\}.$$

EXAMPLES. XVII.

Solve the following equations:

(1) $x^8 - 2x^4 \cos 60^\circ + 1 = 0$.

(2) $x^{10} - 2x^5 \cos 10^\circ + 1 = 0$.

(3) $x^{12} - 2x^6 \cos \frac{2}{3}\pi + 1 = 0$.

(4) $x^{10} + \sqrt{3}x^5 + 1 = 0$.

(5) Write down the factors of $x^{2n} - 2x^n y^n \cos\alpha + y^{2n}$.

*MISCELLANEOUS EXAMPLES. XVIII.

(1) Prove that if $\cos\theta$ and $\sin\theta$ be defined by the equations $2\cos\theta = a^{\theta} + a^{-\theta}$, $2i\sin\theta = a^{\theta} - a^{-\theta}$, then $\sin\theta$ and $\cos\theta$ satisfy the fundamental conditions

$$\cos^2\theta + \sin^2\theta = 1, \quad \cos\theta = \cos(-\theta), \quad \sin\theta = -\sin(-\theta).$$

(2) Prove that if a degree is the unit of angular measurement $2\cos x = k^x + k^{-x}$ where $k^{180} = e^{i\pi}$.

(3) Assuming De Moivre's theorem, prove that

$$\frac{i\sin x^0 - (1 - \cos x^0)}{x} = \log k + \frac{x}{\underline{|2}}(\log k)^2 + \text{etc},$$

where $k = \cos 1^0 + i\sin 1^0 = e^{\frac{i\pi}{180}}$.

(4) Prove that if two right angles be taken as the unit of angle, the exponential values of $\cos x$ and $\sin x$ are $\frac{1}{2}(e^{i\pi x} + e^{-i\pi x})$ and $-\frac{1}{2}i(e^{i\pi x} - e^{-i\pi x})$.

(5) Assuming that $e^{ira} = 1$ where r is an integer, prove that $e^{i\theta}$ is a periodic function of θ.

(6) Assuming that $e^{i\theta}$ is a periodic function of θ, and that the period is 2π, prove that $e^{2ir\pi} = 1$ where r is an integer.

(7) Prove that $(ib + c^i)^a = r^a(\cos a\theta + i\sin a\theta)$ where

$$r^2 = 1 + b^2 + 2b\sin\log c, \quad \tan\theta = \frac{b + \sin\log c}{\cos\log c}.$$

(8) If $\log(1 + \cos 2\theta + i\sin 2\theta) = A + iB$, then

$$A = \log 2 + \log(\cos\theta).$$

(9) Prove that

$$e^x - 2\cos\theta + e^{-x} = 4\sin\tfrac{1}{2}(\theta + ix)\sin\tfrac{1}{2}(\theta - ix).$$

(10) If $\cos^{-1}(\alpha + i\beta) = A + iB$, prove that

$$\frac{\alpha^2}{\cos^2 A} - \frac{\beta^2}{\sin^2 A} = 1. \text{ and } \frac{\alpha^2}{(e^B + e^{-B})^2} + \frac{\beta^2}{(e^B - e^{-B})^2} = \tfrac{1}{4}.$$

(11) Prove that $\log_e(-1) = i(2n+1)\pi$.

(12) Prove that

$$\log(x+iy) = \tfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}.$$

Hence prove that

$$\tan^{-1}\frac{x\sin\theta}{1+x\cos\theta} = x\sin\theta - \frac{x^2}{2}\sin 2\theta + \frac{x^3}{3}\sin 3\theta - $$

(13) f $\phi(\alpha)$ is such a function of α that

$$\phi(\alpha)\times\phi(\beta) = \phi(\alpha+\beta)$$

for all values of α and β, prove that $\phi(\alpha) = \{\phi(1)\}^\alpha$ for all rational values of α. Show that $\cos\alpha + i\sin\alpha$ is a form of $\phi(\alpha)$ which satisfies the preceding equation, and deduce De Moivre's theorem.

(14) Prove that

$$\tan^{-1}(\cos\alpha + i\sin\alpha) = (n+\tfrac{1}{4})\pi + i\log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\alpha).$$

(15) Prove that

$$\frac{\pi}{4} = 4\tan^{-1}\tfrac{1}{5} - \tan^{-1}\tfrac{1}{70} + \tan^{-1}\tfrac{1}{99},$$

and apply the result to find the value of π to 5 places of decimals.

(16) Find the number of radians in the least angle whose tangent is $\frac{1}{10}$; also the number of degrees in the least angle whose tangent is 10.

(17) Prove that the general value of $e^{i\theta}$ is

$$\cos(1+2r\pi)\theta + i\sin(1+2r\pi)\theta$$

where r is an integer.

(18) *Defining* $\cos\theta$ as the real part and $i\sin\theta$ as the imaginary part of $e^{i\theta}$, prove

$\cos\theta = \cos(-\theta)$, $\sin\theta = -\sin(-\theta)$, $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$.

(19) If the functions ϕ and ψ are defined by the equations

$$\phi(x+y) = \phi(x)\,\phi(y) - \psi(x)\,\psi(y),$$
$$\psi(x+y) = \phi(x)\,\psi(y) + \phi(y)\,\psi(x),$$

prove that $$\{\phi(x)^2 + \psi(x)^2\}^{\frac{1}{x}} = \{\phi(y)^2 + \psi(y)^2\}^{\frac{1}{y}}.$$

CHAPTER IV

PROOFS WITHOUT THE USE OF $\sqrt{-1}$

39. In this Chapter we shall give proofs of most of the preceding results by methods which do not involve the use of $\sqrt{-1}$.

The student must not on this account suppose that the validity of results obtained by the aid of $\sqrt{-1}$ is doubtful. We shall make some remarks on this point later on.

40. *To prove, when* n *is a positive integer,*

$$\cos n\alpha = \cos^n\alpha - \frac{n(n-1)}{1 \cdot 2}\cos^{n-2}\alpha \cdot \sin^2\alpha$$
$$+ \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}\cos^{n-4}\alpha \cdot \sin^4\alpha - \text{etc.},$$

and

$$\sin n\alpha = n\cos^{n-1}\alpha \cdot \sin\alpha - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}\cos^{n-3}\alpha \cdot \sin^3\alpha$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}\cos^{n-5}\alpha \cdot \sin^5\alpha - \text{etc.}$$

These formulæ may be proved by **induction,** thus:

Assuming that the above statements are true for a certain value of n, we can prove that they must also be true when $n+1$ is written for n.

Since $\cos(n+1)a = \cos na \,.\, \cos a - \sin na \,.\, \sin a$ [E. 154]; in this, substitute for $\cos na$ and $\sin na$ those values given above, and we obtain

$$\cos(n+1)a = \cos^{n+1}a - \left\{\frac{n(n-1)}{1\,.\,2} + n\right\}\cos^{n-1}a\,.\,\sin^2 a$$

$$+\left\{\frac{n(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4} + \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\right\}\cos^{n-3}a\,.\,\sin^4 a - \text{etc.}$$

The coefficient of $\cos^{n-r}a\,.\,\sin^{r+1}a$ is

$$(-1)^{\frac{r+1}{2}}\left\{\frac{n(n-1)\ldots(n-r+1)(n-r)}{\underline{|r+1}} + \frac{n(n-1)\ldots(n-r+1)}{\underline{|r}}\right\}$$

$$=(-1)^{\frac{r+1}{2}}\left\{\frac{(n+1)\,n\,(n-1)\ldots(n-r+1)}{\underline{|r+1}}\right\}.$$

Therefore $\quad \cos(n+1)a = \cos^{n+1}a - \dfrac{(n+1)\,.\,n}{1\,.\,2}\cos^{n-1}a\,.\,\sin^2 a$

$$+\frac{(n+1)\,n\,(n-1)(n-2)}{1\,.\,2\,.\,3\,.\,4}\cos^{n-3}a\sin^4 a - \text{etc.}$$

A similar result will hold good for $\sin(n+1)a$

Thus, if the formulæ are true when n is any whole number, they are true when $n+1$ is substituted for n;

But they are true when $n=1$ and when $n=2$;

Therefore they are true when n is any whole number.

41. *To prove* $\cos\theta = 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \text{etc.},$

and $\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \text{etc.}$

In the formula

$$\cos n\alpha = \cos^n\alpha - \frac{n(n-1)}{1 \cdot 2}\cos^{n-2}\alpha \cdot \sin^2\alpha$$
$$+ \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}\cos^{n-4}\alpha \cdot \sin^4\alpha + \text{etc.}$$

write θ for $n\alpha$, and let n be increased without limit while θ remains unchanged. Then since $\alpha = \frac{\theta}{n}$, α must be diminished without limit. We may write the above in the following form,

$$\cos\theta = \left(\cos\frac{\theta}{n}\right)^n - \frac{\theta(\theta-\alpha)}{1 \cdot 2}\left(\cos\frac{\theta}{n}\right)^{n-2}\left(\frac{\sin\alpha}{\alpha}\right)^2$$
$$+ \frac{\theta(\theta-\alpha)(\theta-2\alpha)(\theta-3\alpha)}{1 \cdot 2 \cdot 3 \cdot 4}\left(\cos\frac{\theta}{n}\right)^{n-4}\left(\frac{\sin\alpha}{\alpha}\right)^4 - \text{etc}$$

When n is infinitely increased, the limit of $\left(\cos\frac{\theta}{n}\right)^r$ is 1, since r is not greater than n; [Art. 9.]

the limit of $\frac{\theta(\theta-\alpha)(\theta-2\alpha)\ldots(\theta-\overline{r-1}\alpha)}{\underline{|r}}$ is $\frac{\theta^r}{\underline{|r}}$ [Art. 8];

and the limit of $\left(\frac{\sin\frac{\theta}{n}}{\frac{\theta}{n}}\right)^r$ is 1, since r is not greater than n.

[Art. 10.]

Therefore, by proceeding to the limit, we obtain

$$\cos\theta = 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \text{etc.}$$

Similarly, the expansion for $\sin n\alpha$ may be written

$$\sin\theta = \theta \cdot \left(\cos\frac{\theta}{n}\right)^{n-1}\left(\frac{\sin\alpha}{\alpha}\right) \cdot$$
$$- \frac{\theta(\theta-\alpha)(\theta-2\alpha)}{1 \cdot 2 \cdot 3}\left(\cos\frac{\theta}{n}\right)^{n-3}\left(\frac{\sin\alpha}{\alpha}\right)^3 + \text{etc.}$$

By proceeding to the limit, we obtain as before

$$\sin\theta = \theta - \frac{\theta^3}{\lfloor 3} + \frac{\theta^5}{\lfloor 5} - \text{etc}$$

42. In the result $\sin\alpha = \alpha - \frac{\alpha^3}{\lfloor 3} + \frac{\alpha^5}{\lfloor 5} - \frac{\alpha^7}{\lfloor 7} + \text{etc.}$ the series is convergent for *all* values of α.

[For the ratio of any term to the preceding is $\frac{\alpha^2}{n(n+1)}$; and whatever be the value of α, by taking n large enough this fraction can be made less than some quantity which is itself less than unity.]

In the proof of Art. 41 no limit was put upon the value of the angle. Therefore the result is arithmetically intelligible and true for all values of α.

Therefore the series $\alpha - \frac{\alpha^3}{\lfloor 3} + \frac{\alpha^5}{\lfloor 5} - \frac{\alpha^7}{\lfloor 7} + \text{etc.}$, which is equal to $\sin\alpha$ for all values of α, must be **periodic**. [Art. 27.]

43. A series in *ascending* powers of a quantity (α) is chiefly useful when α is small, for the smaller the quantity α, the greater is the relative importance of the *earlier* terms of the series. Also, the sine (and the cosine) of an angle of any magnitude may be expressed in terms of the sine or cosine of some angle less than $\frac{1}{4}\pi$.

Hence, the above series are never used in numerical calculations except for values of α less than $\frac{1}{4}\pi$.

44. We have

$$\sin a = a - \left(\frac{a^3}{\lfloor 3} - \frac{a^5}{\lfloor 5}\right) - \left(\frac{a^7}{\lfloor 7} - \frac{a^9}{\lfloor 9}\right) - \text{etc.}$$

$$= \left(a - \frac{a^3}{\lfloor 3}\right) + \left(\frac{a^5}{\lfloor 5} - \frac{a^7}{\lfloor 7}\right) + \left(\frac{a^9}{\lfloor 9} - \frac{a^{11}}{\lfloor 11}\right) + \text{etc.}$$

Each of the above brackets is *positive* (provided a^2 is not greater than 6 and therefore, *a fortiori*, if a is less than 1). Therefore $\sin a$ is less than a and greater than $a - \frac{1}{6}a^3$.

Again

$(\sin a - a)$ is *negative* and $= -\frac{1}{6}a^3 +$ (a positive quantity),

$\sin a - (a - \frac{1}{6}a^3)$ is *positive* and $= \frac{1}{120}a^5 -$ (a positive quantity). Therefore the difference between a and $\sin a$ is less than $\frac{1}{6}a^3$; the difference between $\sin a$ and $a - \frac{1}{6}a^3$ is less than $\frac{1}{120}a^5$.

Example. If $a = \frac{1}{10}$ (of a radian), the difference between a and $\sin a$ is less than $\frac{1}{6} . 10^{-3}$, *i.e.* less than a six-hundredth part of a. The difference between $\sin a$ and $a - \frac{1}{6}a^3$ is less than $\frac{1}{120} \times 10^{-5}$ which is less than a millionth part of a.

45. The following results may be proved in a similar manner.

The difference between 1 and $\cos a$ is less than $\frac{1}{2}a^2$,

. .. . .. $\cos a$ and $(1 - \frac{1}{2}a^2)$ ..... $\frac{1}{24}a^4$.

Example. *Find the limiting value of* $\frac{\sin^2 ma}{1 - \cos na}$ *when* a *is infinitely diminished.*

For $\sin ma$, write $ma - R_1a^3$; for $\cos na$, write $1 - \frac{1}{2}n^2a^2 + R_2a^4$. We know that R_1 is less than $\frac{1}{6}m^3$ and that R_2 is less than $\frac{1}{24}n^4$. Then

$$\frac{\sin^2 ma}{1 - \cos na} = \frac{(ma - R_1a^3)^2}{\frac{1}{2}n^2a^2 - R_2a^4} = \frac{(m - R_1a^2)^2}{\frac{1}{2}n^2 - R_2a^2};$$

hence, when a is infinitely diminished the required limit is $\frac{2m^2}{n^2}$.

EXAMPLES. XIX.

(1) Prove that when a is not large, the difference between $(a - \frac{1}{6}a^3 + \frac{1}{120}a^5)$ and $\sin a$ is less than $\frac{1}{5040}a^7$.

(2) Prove that when a is not large, the difference between $(1 - \frac{1}{2}a^2 + \frac{1}{24}a^4)$ and $\cos a$ is less than $\frac{1}{720}a^6$.

(3) Prove that $\sin \dfrac{18^0}{\pi} = \cdot 099833....$

(4) Prove that the value of $\sin 1^0$ coincides with that of the circular measure of 1^0 at least as far as five places of decimals.

(5) Solve the equation $\sin(\frac{1}{4}\pi + \theta) = \cdot 71$, neglecting θ^3 and higher powers of θ.

(6) Given that $\sin 1' = 0002909$, calculate approximately the value of π.

(7) Given $\dfrac{\sin\theta}{\theta} = \dfrac{1013}{1014}$, prove $\theta = 4^0 24'$ nearly.

(8) Find the value of $\dfrac{\sin^2 n\theta - \sin^2 m\theta}{1 - \cos p\theta}$ when $\theta = 0$

(9) Evaluate $\dfrac{\sin^2\sqrt{mn}\theta - \sin m\theta \, . \, \sin n\theta}{(1 - \cos m\theta)(1 - \cos n\theta)}$ when $\theta = 0$.

(10) Find the limit of $\theta^{-4}\left\{\dfrac{\sin\theta}{\theta} + \dfrac{e^\theta - e^{-\theta}}{2\theta} - 2\right\}$ when θ is infinitely diminished.

(11) Prove that (eight times the chord of half a small circular arc minus the chord of the whole arc) divided by three, is equal to the length of the arc, nearly.

(12) Prove by induction

$$\tan n\theta = \frac{n\tan\theta - \dfrac{n(n-1)(n-2)}{1 \, . \, 2 \, . \, 3}\tan^3\theta + \ldots\ldots}{1 - \dfrac{n(n-1)}{1 \, . \, 2}\tan^2\theta + \quad \ldots}.$$

(13) $\sin(\theta + h) = \sin\theta + h\cos\theta - \dfrac{h^2}{\underline{|2}}\sin\theta - \dfrac{h^3}{\underline{|3}}\cos\theta + \text{etc.}$

THE HYPERBOLIC FUNCTIONS.

46. The following notation will be found convenient.

Def. **cosh** x stands for $\dfrac{e^x + e^{-x}}{2}$ and **sinh** x for $\dfrac{e^x - e^{-x}}{2}$.

cosh x and **sinh** x are abbreviations respectively for the words **hyperbolic cosine of** x and **hyperbolic sine of** x.

We shall use this notation $\cosh^2 x$ for $(\cosh x)^2$, etc.

EXAMPLES. XX.

Prove the following statements: [Compare Examples XII.]

(1) $\cosh^2 x - \sinh^2 x = 1.$

(2) $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x.$

(3) $\cosh 3x = 4\cosh^3 x - 3\cosh x$ (4) $\cosh x = \cosh(-x).$

(5) $\sinh 3x = 3\sinh x + 4\sinh^3 x.$ (6) $\sinh x = -\sinh(-x).$

(7) $\cosh(x+y) \,.\, \cosh(x-y) = \cosh^2 x + \sinh^2 y$
$= \cosh^2 y + \sinh^2 x.$

(8) $2\cosh nx \,.\, \cosh x = \cosh(n+1)x + \cosh(n-1)x$

(9) $2\sinh nx \,.\, 2^2\sinh^2 x = 2\sinh(n+2)x - 4\sinh nx$
$+ 2\sinh(n-2)x.$

(10) $2\cosh nx \,.\, 2^2\sinh^2 x = 2\cosh(n+2)x - 4\cosh nx$
$+ 2\cosh(n-2)x.$

(11) $\cosh nx - \cos n\alpha = 2\cosh(n-1)x\,\{\cosh x - \cos\alpha\}$
$+ 2\cos\alpha\,\{\cosh(n-1)x - \cos(n-1)\alpha\} - \{\cosh(n-2)x - \cos(n-2)\alpha\}.$

(12) $\cosh 2x - \cos 2\theta = 2(\sin^2\theta\cosh^2 x + \cos^2\theta\sinh^2 x).$

(13) $\cosh y \cosh z \sinh(y-z) + \cosh z \cosh x \sinh(z-x)$
$+ \cosh x \cosh y \sinh(x-y) + \sinh(y-z)\sinh(z-x)\sinh(x-y) = 0.$

(14) $\cosh 3x + \cosh 5x + \cosh 7x + \cosh 15x$
$= 4\cosh 4x \cosh 5x \cosh 6x.$

47. Similarly, $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ is called the **hyperbolic tangent** of x, and is written **tanh** x, and so on for other hyperbolic functions corresponding to the other circular functions.

48. Since

$$\sin x = \frac{1}{2i}(e^{ix} - e^{-ix}) \text{ and } \cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}),$$

therefore

$$\sin ix = \frac{1}{2i}(e^{-x} - e^{x}) = \tfrac{1}{2}i(e^{x} - e^{-x})$$

$$= i \sinh x,$$

and

$$\cos ix = \tfrac{1}{2}(e^{-x} + e^{x})$$

$$= \cosh x.$$

*49. It should be noticed that when a complete interpretation has been given to the symbol i, it may be possible to interpret the symbols $\sin ix$ and e^{ix}, and then it will be seen how it is that $\sin ix$ and $i \sinh x$ indicate the same thing. For the present it is sufficient to notice that one result of these equalities is that the expressions $i \sinh x$ and $\cosh x$ are combined by the same laws as those by which $\sin ix$ and $\cos ix$ are combined.

And therefore it follows from Art. 48 that corresponding to any theorem, involving $\cos\theta$ and $\sin\theta$, which is true for all values of θ, there is a theorem involving $\cosh x$ and $i \sinh x$ in precisely the same form.

Example i. Since $\cos 2\theta = \cos^2\theta - \sin^2\theta$,
therefore $\cosh 2x = \cosh^2 x + \sinh^2 x$.

Example ii. Since $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$,
therefore $i \sinh 3x = 3i \sinh x - 4i^3 \sinh^3 x$,
or $\sinh 3x = 3\sinh x + 4\sinh^3 x$.

Example iii. *Prove that*

$$\sin(\alpha + i\beta) = \frac{e^{\beta} + e^{-\beta}}{2}\sin\alpha + i\frac{e^{\beta} - e^{-\beta}}{2}\cos\alpha$$

$$\begin{aligned}\sin(\alpha + i\beta) &= \sin\alpha\cos i\beta + \cos\alpha\sin i\beta\\ &= \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta. \qquad \text{Q. E. D.}\end{aligned}$$

Example iv. *Prove that if*

$$x + iy = \kappa\cos(\theta - i\eta),$$

then
$$\frac{x^2}{\kappa^2\cosh^2\eta} + \frac{y^2}{\kappa^2\sinh^2\eta} = 1,$$

and
$$\frac{x^2}{\kappa^2\cos^2\theta} - \frac{y^2}{\kappa^2\sin^2\theta} = 1$$

Since
$$\begin{aligned}x + iy &= \kappa\cos\theta\cos i\eta + \kappa\sin\theta\sin i\eta\\ &= \kappa\cos\theta\cosh\eta + i\kappa\sin\theta\sinh\eta,\end{aligned}$$

equating the real and imaginary parts

we have
$$x = \kappa\cos\theta\cosh\eta, \quad y = \kappa\sin\theta\sinh\eta,$$

or
$$\frac{x}{\kappa\cos\theta} = \cosh\eta, \quad \frac{y}{\kappa\sin\theta} = \sinh\eta,$$

wherefore
$$\frac{x^2}{\kappa^2\cos^2\theta} - \frac{y^2}{\kappa^2\sin^2\theta} = \cosh^2\eta - \sinh^2\eta = 1$$

Similarly
$$\frac{x}{\kappa\cosh\eta} = \cos\theta, \quad \frac{y}{\kappa\sinh\eta} = \sin\theta,$$

wherefore
$$\frac{x^2}{\kappa^2\cosh^2\eta} + \frac{y^2}{\kappa^2\sinh^2\eta} = \cos^2\theta + \sin^2\theta = 1$$

Q. E. D.

It will be seen by those acquainted with plane coordinate Geometry that for given values of θ and η these equations represent a confocal ellipse and hyperbola, such that sech η is the eccentricity of the ellipse and sec θ the eccentricity of the hyperbola.

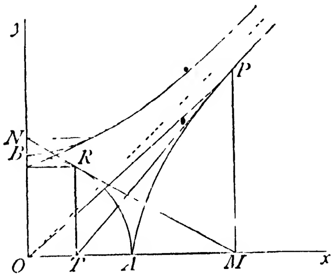
*50. The Geometrical interpretation of the hyperbolic functions is as follows:

Draw the rectangular hyperbola

$$x^2 - y^2 = a^2 \qquad \ldots\ldots(\text{i}),$$

its conjugate
$$y^2 - x^2 = a^2 \qquad \ldots\ldots(\text{ii}),$$

and the inscribed circle
$$x^2 + y^2 = a^2. \qquad \ldots\ldots(\text{iii}).$$



In the Figure, ARB is a quadrant of the circle (iii), AP an arc of the equilateral hyperbola (i) of which OA and OB are the semi-axes.

Then if x, y are the coordinates of a point P on (i) we may put

$$x = OM = a \cosh u,$$

$$y = MP = a \sinh u,$$

since

$$a^2 \cosh^2 u - a^2 \sinh^2 u = a^2.$$

From M draw MR to touch the circle (iii) and let θ be the circular measure of MOR, then

$$x = a \sec \theta = a \cosh u,$$

$$y = a \tan \theta = a \sinh u.$$

The sectorial area $AOR = \frac{1}{2}a^2\theta$, and it can be proved [see Greenhill's *Differential and Integral Calculus*, Art. 56] that the sectorial area $OAP = \frac{1}{2}a^2 u$.

*51. On the following pages we give tables of the value of each circular and hyperbolic function direct and inverse each in terms of the others.

The student is recommended to verify these tables for himself and also to deduce [by Art. 49] the hyperbolic tables from the circular and conversely.

TABLE I.—GIVING ANY DIRECT CIRCULAR FUNCTION IN TERMS OF ANY OTHER.

	Sine.	Cosine.	Tangent.	Cotangent.	Secant.	Cosecant
$\sin\theta=$	$\sin\theta$	$\sqrt{(1-\cos^2\theta)}$	$\frac{\tan\theta}{\sqrt{(1+\tan^2\theta)}}$	$\frac{1}{\sqrt{(\cot^2\theta+1)}}$	$\frac{\sqrt{(\sec^2\theta-1)}}{\sec\theta}$	$\frac{1}{\operatorname{cosec}\theta}$
$\cos\theta=$	$\sqrt{(1-\sin^2\theta)}$	$\cos\theta$	$\frac{1}{\sqrt{(1+\tan^2\theta)}}$	$\frac{\cot\theta}{\sqrt{(\cot^2\theta+1)}}$	$\frac{1}{\sec\theta}$	$\frac{\sqrt{(\operatorname{cosec}^2\theta-1)}}{\operatorname{cosec}\theta}$
$\tan\theta=$	$\frac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}$	$\frac{\sqrt{(1-\cos^2\theta)}}{\cos\theta}$	$\tan\theta$	$\frac{1}{\cot\theta}$	$\sqrt{(\sec^2\theta-1)}$	$\frac{1}{\sqrt{(\operatorname{cosec}^2\theta-1)}}$
$\cot\theta=$	$\frac{\sqrt{(1-\sin^2\theta)}}{\sin\theta}$	$\frac{\cos\theta}{\sqrt{(1-\cos^2\theta)}}$	$\frac{1}{\tan\theta}$	$\cot\theta$	$\frac{1}{\sqrt{(\sec^2\theta-1)}}$	$\sqrt{(\operatorname{cosec}^2\theta-1)}$
$\sec\theta=$	$\frac{1}{\sqrt{(1-\sin^2\theta)}}$	$\frac{1}{\cos\theta}$	$\sqrt{(1+\tan^2\theta)}$	$\frac{\sqrt{(\cot^2\theta+1)}}{\cot\theta}$	$\sec\theta$	$\frac{\operatorname{cosec}\theta}{\sqrt{(\operatorname{cosec}^2\theta-1)}}$
$\operatorname{cosec}\theta=$	$\frac{1}{\sin\theta}$	$\frac{1}{\sqrt{(1-\cos^2\theta)}}$	$\frac{\sqrt{(1+\tan^2\theta)}}{\tan\theta}$	$\sqrt{(\cot^2\theta+1)}$	$\frac{\sec\theta}{\sqrt{(\sec^2\theta-1)}}$	$\operatorname{cosec}\theta$

TABLE II.—GIVING ANY DIRECT HYPERBOLIC FUNCTION IN TERMS OF ANY OTHER.

	Sinh x.	Cosh x.	Tanh x.	Coth x.	Sech x.	Cosech x.
$\sinh x =$	$\sinh x$	$\sqrt{(\cosh^2 x - 1)}$	$\dfrac{\tanh x}{\sqrt{(1-\tanh^2 x)}}$	$\dfrac{1}{\sqrt{(\coth^2 x - 1)}}$	$\dfrac{\sqrt{(1-\operatorname{sech}^2 x)}}{\operatorname{sech} x}$	$\dfrac{1}{\operatorname{cosech} x}$
$\cosh x =$	$\sqrt{(1+\sinh^2 x)}$	$\cosh x$	$\dfrac{1}{\sqrt{(1-\tanh^2 x)}}$	$\dfrac{\coth x}{\sqrt{(\coth^2 x - 1)}}$	$\dfrac{1}{\operatorname{sech} x}$	$\dfrac{\sqrt{(\operatorname{cosech}^2 x + 1)}}{\operatorname{cosech} x}$
$\tanh x =$	$\dfrac{\sinh x}{\sqrt{(1+\sinh^2 x)}}$	$\dfrac{\sqrt{(\cosh^2 x - 1)}}{\cosh x}$	$\tanh x$	$\dfrac{1}{\coth x}$	$\sqrt{(1-\operatorname{sech}^2 x)}$	$\dfrac{1}{\sqrt{(\operatorname{cosech}^2 x + 1)}}$
$\coth x =$	$\dfrac{\sqrt{(1+\sinh^2 x)}}{\sinh x}$	$\dfrac{\cosh x}{\sqrt{(\cosh^2 x - 1)}}$	$\dfrac{1}{\tanh x}$	$\coth x$	$\dfrac{1}{\sqrt{(1-\operatorname{sech}^2 x)}}$	$\sqrt{(\operatorname{cosech}^2 x + 1)}$
$\operatorname{sech} x =$	$\dfrac{1}{\sqrt{(1+\sinh^2 x)}}$	$\dfrac{1}{\cosh x}$	$\sqrt{(1-\tanh^2 x)}$	$\dfrac{\sqrt{(\coth^2 x - 1)}}{\coth x}$	$\operatorname{sech} x$	$\dfrac{\operatorname{cosech} x}{\sqrt{(\operatorname{cosech}^2 x + 1)}}$
$\operatorname{cosech} x =$	$\dfrac{1}{\sinh x}$	$\dfrac{1}{\sqrt{(\cosh^2 x - 1)}}$	$\dfrac{\sqrt{(1-\tanh^2 x)}}{\tanh x}$	$\sqrt{(\coth^2 x - 1)}$	$\dfrac{\operatorname{sech} x}{\sqrt{(1-\operatorname{sech}^2 x)}}$	$\operatorname{cosech} x$

TABLE III.—GIVING ANY INVERSE CIRCULAR FUNCTION IN TERMS OF ANY OTHER.

	Sin^{-1}.	Cos^{-1}.	Tan^{-1}.	Cot^{-1}.	Sec^{-1}.	Cosec^{-1}.
$\sin^{-1}x=$	$\sin^{-1}x$	$\cos^{-1}\sqrt{(1-x^2)}$	$\tan^{-1}\frac{x}{\sqrt{(1-x^2)}}$	$\cot^{-1}\frac{\sqrt{(1-x^2)}}{x}$	$\sec^{-1}\frac{1}{\sqrt{(1-x^2)}}$	$\operatorname{cosec}^{-1}\frac{1}{x}$
$\cos^{-1}x=$	$\sin^{-1}\sqrt{(1-x^2)}$	$\cos^{-1}x$	$\tan^{-1}\frac{\sqrt{(1-x^2)}}{x}$	$\cot^{-1}\frac{x}{\sqrt{(1-x^2)}}$	$\sec^{-1}\frac{1}{x}$	$\operatorname{cosec}^{-1}\frac{1}{\sqrt{(1-x^2)}}$
$\tan^{-1}x=$	$\sin^{-1}\frac{x}{\sqrt{(1+x^2)}}$	$\cos^{-1}\frac{1}{\sqrt{(1+x^2)}}$	$\tan^{-1}x$	$\cot^{-1}\frac{1}{x}$	$\sec^{-1}\sqrt{(1+x^2)}$	$\operatorname{cosec}^{-1}\frac{\sqrt{(1+x^2)}}{x}$
$\cot^{-1}x=$	$\sin^{-1}\frac{1}{\sqrt{(x^2+1)}}$	$\cos^{-1}\frac{x}{\sqrt{(x^2+1)}}$	$\tan^{-1}\frac{1}{x}$	$\cot^{-1}x$	$\sec^{-1}\frac{\sqrt{(x^2+1)}}{x}$	$\operatorname{cosec}^{-1}\sqrt{(x^2+1)}$
$\sec^{-1}x=$	$\sin^{-1}\frac{\sqrt{(x^2-1)}}{x}$	$\cos^{-1}\frac{1}{x}$	$\tan^{-1}(\sqrt{x^2-1})$	$\cot^{-1}\frac{1}{\sqrt{(x^2-1)}}$	$\sec^{-1}x$	$\operatorname{cosec}^{-1}\frac{x}{\sqrt{(x^2-1)}}$
$\operatorname{cosec}^{-1}x=$	$\sin^{-1}\frac{1}{x}$	$\cos^{-1}\frac{\sqrt{(x^2-1)}}{x}$	$\tan^{-1}\frac{1}{\sqrt{(x^2-1)}}$	$\cot^{-1}\sqrt{(x^2-1)}$	$\sec^{-1}\frac{x}{\sqrt{(x^2-1)}}$	$\operatorname{cosec}^{-1}x$

TABLE IV.—GIVING ANY INVERSE HYPERBOLIC FUNCTION IN TERMS OF ANY OTHER.

	$\mathrm{Sinh}^{-1} x.$	$\mathrm{Cosh}^{-1} x.$	$\mathrm{Tanh}^{-1} x.$	$\mathrm{Coth}^{-1} x.$	$\mathrm{Sech}^{-1} x.$	$\mathrm{Cosech}^{-1} x.$
$\sinh^{-1} x =$	$\sinh^{-1} x$	$\cosh^{-1} \sqrt{1+x^2}$	$\tanh^{-1} \frac{x}{\sqrt{1+x^2}}$	$\coth^{-1} \frac{\sqrt{1+x^2}}{x}$	$\mathrm{sech}^{-1} \frac{1}{\sqrt{1+x^2}}$	$\mathrm{cosech}^{-1} \frac{1}{x}$
$\cosh^{-1} x =$	$\sinh^{-1} \sqrt{x^2-1}$	$\cosh^{-1} x$	$\tanh^{-1} \frac{\sqrt{x^2-1}}{x}$	$\coth^{-1} \frac{x}{\sqrt{x^2-1}}$	$\mathrm{sech}^{-1} \frac{1}{x}$	$\mathrm{cosech}^{-1} \frac{1}{\sqrt{x^2-1}}$
$\tanh^{-1} x =$	$\sinh^{-1} \frac{x}{\sqrt{1-x^2}}$	$\cosh^{-1} \frac{1}{\sqrt{1-x^2}}$	$\tanh^{-1} x$	$\coth^{-1} \frac{1}{x}$	$\mathrm{sech}^{-1} \sqrt{1-x^2}$	$\mathrm{cosech}^{-1} \frac{\sqrt{1-x^2}}{x}$
$\coth^{-1} x =$	$\sinh^{-1} \frac{1}{\sqrt{x^2-1}}$	$\cosh^{-1} \frac{x}{\sqrt{x^2-1}}$	$\tanh^{-1} \frac{1}{x}$	$\coth^{-1} x$	$\mathrm{sech}^{-1} \frac{\sqrt{x^2-1}}{x}$	$\mathrm{cosech}^{-1} \sqrt{x^2-1}$
$\mathrm{sech}^{-1} x =$	$\sinh^{-1} \frac{\sqrt{1-x^2}}{x}$	$\cosh^{-1} \frac{1}{x}$	$\tanh^{-1} \sqrt{1-x^2}$	$\coth^{-1} \frac{1}{\sqrt{1-x^2}}$	$\mathrm{sech}^{-1} x$	$\mathrm{cosech}^{-1} \frac{x}{\sqrt{1-x^2}}$
$\mathrm{cosech}^{-1} x =$	$\sinh^{-1} \frac{1}{x}$	$\cosh^{-1} \frac{\sqrt{x^2+1}}{x}$	$\tanh^{-1} \sqrt{\frac{1}{(1+x^2)}}$	$\coth^{-1} \sqrt{1+x^2}$	$\mathrm{sech}^{-1} \frac{x}{\sqrt{x^2+1}}$	$\mathrm{cosech}^{-1} x$

**Example* I. *Prove that* $\tanh^{-1} x = \sinh^{-1} \dfrac{x}{\sqrt{\{1-x^2\}}}$.

Here, $\tanh^{-1}x$ *stands for* **the number whose hyperbolic tangent is** x.

Let $a = \tanh^{-1} x$; then $\tanh a = x = \dfrac{e^a - e^{-a}}{e^a + e^{-a}}$,

also,
$$\tfrac{1}{2}(e^a - e^{-a}) = \frac{\dfrac{e^a - e^{-a}}{e^a + e^{-a}}}{\sqrt{\left\{1 - \left(\dfrac{e^a - e^{-a}}{e^a + e^{-a}}\right)^2\right\}}},$$

or,
$$\sinh a = \frac{\tanh a}{\sqrt{\{1 - \tanh^2 a\}}},$$

or,
$$a = \sinh^{-1} \frac{\tanh a}{\sqrt{\{1 - \tanh^2 a\}}},$$

or,
$$\tanh^{-1} x = \sinh^{-1} \frac{x}{\sqrt{\{1 - x^2\}}}. \qquad \text{Q.E.D.}$$

Example II. *Sum the series*

$$1 + x\cosh\theta + x^2\cosh 2\theta + \ldots + x^{n-1}\cosh(n-1)\theta.$$

Let C be the sum of the series, and let

$$S = x\sinh\theta + x^2\sinh 2\theta + \ldots + x^{n-1}\sinh(n-1)\theta,$$

then
$$C + S = 1 + xe^{\theta} + x^2e^{2\theta} + \ldots + x^{n-1}e^{(n-1)\theta},$$

also
$$C - S = 1 + xe^{-\theta} + x^2e^{-2\theta} + \ldots + x^{n-1}e^{-(n-1)\theta}.$$

Hence
$$C + S = \frac{1 - x^n e^{n\theta}}{1 - xe^{\theta}}, \quad C - S = \frac{1 - x^n e^{-n\theta}}{1 - xe^{-\theta}},$$

$$\therefore\ 2C =$$
$$\frac{2 + x^{n+1}\{e^{(n-1)\theta} + e^{-(n-1)\theta}\} - x\{e^{+\theta} + e^{-\theta}\} - x^n\{e^{n\theta} + e^{-n\theta}\}}{1 - x(e^{\theta} + e^{-\theta}) + x^2};$$

$$\therefore\ C = \frac{1 + x^{n+1}\cosh(n-1)\theta - x\cosh\theta - x^n\cosh n\theta}{1 - 2x\cosh\theta + x^2}.$$

[Compare Ex. 1, p. 82.]

*EXAMPLES. XXI.

Prove the following statements:

(1) $\tanh(x+y) = \dfrac{\tanh x + \tanh y}{1+\tanh x \tanh y}$.

(2) $\sinh^{-1} x = \cosh^{-1}\sqrt{(1+x^2)} = \tanh^{-1} \dfrac{x}{\sqrt{(1+x^2)}}$.

(3) $\tanh^{-1} x + \tanh^{-1} y = \tanh^{-1} \dfrac{x+y}{1+xy}$.

(4) $\sinh^{-1} x = \log\{\sqrt{(1+x^2)}+x\}$.

(5) $\cosh^{-1} x = \log\{\sqrt{(x^2-1)}+x\}$.

(6) $\tanh^{-1} x = \frac{1}{2}\{\log(1+x) - \log(1-x)\}$.

(7) $\dfrac{\tanh^2 x - \sin^2 \alpha}{\tanh^2 x - \sin^2 \beta} = \dfrac{\cot^2\alpha \sinh^2 x - 1}{\cot^2\beta \sinh^2 x - 1} \times \dfrac{\sin^2\alpha}{\sin^2\beta}$.

(8) $\dfrac{\sinh(b-y)\sinh(b-z)}{\sinh(a-y)\sinh(a-z)} \sinh 2(a-b)$

+ two symmetrical expressions in $x, y, z = 0$.

(9) $\log\tan(\alpha + \iota x) = \frac{1}{2}\log\dfrac{\cosh 4x - \cos 4\alpha}{2(\cosh 2x + \cos 2\alpha)^2} + \iota \tan^{-1}\dfrac{\sinh 2x}{\sin 2\alpha}$.

(10) Sum the series

$$\cosh a + \cosh(a+b) + \cosh(a+2b)$$

to n terms, and deduce the sum of the series

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta).$$

(11) Prove that

$$\tan^{-1}\left(\frac{\tan 2\theta + \tanh 2x}{\tan 2\theta - \tanh 2x}\right) + \tan^{-1}\left(\frac{\tan\theta - \tanh x}{\tan\theta + \tanh x}\right)$$
$$= \tan^{-1}(\cot\theta \coth x).$$

(12) $\tanh(\alpha+\beta+\gamma)$

$$= \frac{\tanh\alpha + \tanh\beta + \tanh\gamma + \tanh\alpha\tanh\beta\tanh\gamma}{1 + \tanh\beta\tanh\gamma + \tanh\gamma\tanh\alpha + \tanh\alpha\tanh\beta}.$$

(13) The sum of the series

$$1 + \cosh x + \frac{\cosh 2x}{\lfloor 2} + \frac{\cosh 3x}{\lfloor 3} \quad \text{ad inf. is } e^{\cosh x}\cosh(\sinh x).$$

EXPANSION OF $(2\cos\alpha)^n$ AND OF $(2\sin\alpha)^n$.

52. *To prove, when* n *is a positive integer, that* $2^n \cos^n\alpha$ *can be expressed in terms of* $\cos n\alpha$, $\cos(n-2)\alpha$, etc.; *that* $2^n \cosh^n x$ *can be expressed in terms of* $\cosh nx$, $\cosh(n-2)x$, etc., *and that the two expressions are the* **same** *in form.*

We have, when n is a positive integer,

$$2\cos n\alpha \,.\, 2\cos\alpha = 2\cos(n+1)\alpha + 2\cos(n-1)\alpha. \qquad \text{I.}$$

$$2\cosh nx \,.\, 2\cosh x = 2\cosh(n+1)x + 2\cosh(n-1)x. \qquad \text{II.}$$

In I. put $n=1$, and we obtain

$$2^2\cos^2\alpha = 2\cos 2\alpha + 2.$$

Multiply each side of the result by $2\cos\alpha$, then

$$2^3\cos^3\alpha = 2\cos 2\alpha \,.\, 2\cos\alpha + 4\cos\alpha.$$

But by I. $2\cos 2\alpha \,.\, 2\cos\alpha = 2\cos 3\alpha + 2\cos\alpha$,

hence $$2^3\cos^3\alpha = 2\cos 3\alpha + 6\cos\alpha.$$

Multiplying each side of this result by $2\cos\alpha$, and again making use of I., we have

$$2^4\cos^4\alpha = 2\cos 4\alpha + 8\cos 2\alpha + 6.$$

By multiplying each side of this result by $2\cos\alpha$, and making use of I. on the right-hand side, we can obtain an expression for $2^5\cos^5\alpha$ in the required form. And so on.

By continuing this process we could obtain an expression in the required form for $2^n\cos^n\alpha$ where n is any positive integer.

Again, by making use of II. in the same manner we could obtain an expression in the required form for $2^n\cosh^n x$.

Also, since the *process is the same in each case*, the two resulting expressions are the *same* in form.

53. The expansion of $2^n\cosh^n x$ can be found as follows:

$$2^n\cosh^n x = (e^x + e^{-x})^n = e^{nx} + e^{-nx} + n\left(e^{(n-2)x} + e^{-(n-2)x}\right) + \text{etc.}$$
$$= 2\cosh nx + n\,.\,2\cosh(n-2)x$$
$$+ \frac{n(n-1)}{1\,.\,2}2\cosh(n-4)x + \text{etc.}$$

Therefore also by Art. 52

$$2^n\cos^n\alpha = 2\cos n\alpha + n\,.\,2\cos(n-2)\alpha + \frac{n(n-1)}{1\,.\,2}2\cos(n-4)\alpha + \text{etc.}$$

As in Art. 37, when n is even, the last term does not contain $\cosh x$, and in this term the factor 2 is to be omitted.

*54. *To prove, when* n *is* **odd**, *that* $(-1)^{\frac{n-1}{2}}2^n\sin^n\alpha$ *can be expressed in terms of* $\sin n\alpha$, $\sin(n-2)\alpha$, etc.; *that* $2^n\sinh nx$ *can be expressed in terms of* $\sinh nx$, $\sinh(n-2)x$, etc.; *and that the two expressions are the* **same** *in form.*

We have, when n is a positive integer > 2,

$$2\sin n\alpha\,(-1)\,2^2\sin^2\alpha$$
$$= 2\sin(n+2)\alpha - 4\sin n\alpha + 2\sin(n-2)\alpha. \qquad \text{I.}$$
$$2\sinh nx\,.\,2^2\sinh^2 x$$
$$= 2\sinh(n+2)x - 4\sinh nx + 2\sinh(n-2)x. \qquad \text{II.}$$

We have also

$$(-1)\,2^3\sin^3\alpha = 2\sin 3\alpha - 6\sin\alpha.$$

We proceed as in Art. 52. Multiply each side of this result by $-2^2\sin^2\alpha$, and we obtain by the aid of I.

$$(-1)^2\,2^5\sin^5\alpha = 2\sin 5\alpha - 10\sin 3\alpha + 20\sin\alpha.$$

Multiplying again by $-2^2\sin^2\alpha$, we could obtain by the aid of I. an expression in the required form for $(-1)^3\,2^7\sin^7\alpha$.

By continuing this process we could obtain an expression in the required form for $(-1)^{\frac{n-1}{2}}2^n\sin^n\alpha$, where n is any odd positive integer.

Again, since $2^3 \sinh^3 x = 2 \sinh 3x - 6 \sinh x$, by making use of II. in the same manner we could obtain an expression in the required form for $2^n \sinh^n x$.

And since *the process* on the right hand *is the same* in each case, the resulting expressions are the *same* in form.

*55. We have $2^n \sinh^n x = (e^x - e^{-x})^n$ [n *odd*]

$$= e^{nx} - e^{-nx} - n\left(e^{(n-2)x} - e^{-(n-2)x}\right) + \text{etc.}$$

$$= 2 \sinh nx - n 2 \sinh (n-2) x$$

$$+ \frac{n(n-1)}{1 \cdot 2} 2 \sinh (n-4) x - \text{etc.}$$

Therefore it follows by Art. 54, that when n is odd,

$$(-1)^{\frac{n-1}{2}} 2^n \sin^n a = 2 \sin na - n 2 \sin (n-2) a$$

$$+ \frac{n(n-1)}{1 \cdot 2} 2 \sin (n-4) a - \text{etc.}$$

*56. *To prove that, when* n *is* **even**,

$$(-1)^{\frac{n}{2}} 2^n \sin^n a = 2 \cos na - n 2 \cos (n-2) a$$

$$+ \frac{n(n-1)}{1 \cdot 2} 2 \cos (n-4) a - \text{etc.}$$

We have, when n is a positive integer > 2,

$$2 \cos na \,.\, (-1) 2^2 \sin^2 a = 2 \cos (n+2) a - 4 \cos na + 2 \cos (n-2) a$$

$$2 \cosh nx \,.\, 2^2 \sinh^2 x$$

$$= 2 \cosh (n+2) x - 4 \cosh nx + 2 \cosh (n-2) x.$$

Following the argument of articles 54 and 55, we have, since

$$(-1) 2^2 \sin^2 a = 2 \cos 2a - 2, \text{ and } 2^2 \sinh^2 x = 2 \cosh 2x - 2.$$

And since $2^n \sinh^n x = (e^x - e^{-x})^n$ [n *even*]

$$= 2 \cosh nx - n 2 \cosh (n-2) x + \frac{n(n-1)}{1 \cdot 2} 2 \cosh (n-4) x - \text{etc.}$$

Therefore when n is even $(-1)^{\frac{n}{2}} 2^n \sin^n\alpha$

$$= 2\cos n\alpha - n2\cos(n-2)\alpha + \frac{n(n-1)}{1 \cdot 2} 2\cos(n-4)\alpha - \text{etc.}$$

In the last term the factor 2 must be omitted.

[Cf. Art. 37.]

*57. It must be noticed that in Art. 52 we take a particular formula involving the cosines of multiples of α, and shew that from it a certain expansion of $(2\cos\alpha)^n$ can be deduced; we then shew that a formula of precisely the same shape involving the hyperbolic cosines of multiples of x, is also true, and then we see that the corresponding expansion of $(2\cosh x)^n$ must have *the same form* as that of $(2\cos\alpha)^n$. Thus in Art. 52 we give an *example* of the truth of the general statement made in Art. 49.

Assuming the truth of the theorem of Art. 49 we may say at once: since

$$2^n\cosh^n x = 2\cosh nx + n2\cosh(n-2)x + \text{etc.},$$

therefore, by Art. 49,

$$2^n\cos^n\alpha = 2\cos n\alpha + n2\cos(n-2)\alpha + \text{etc.}$$

*EXAMPLES. XXII.

Prove that

(1) $8\cos^4\theta = \cos 4\theta + 4\cos 2\theta + 3$.

(2) $-64\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$.

(3) $128\sin^8\theta = \cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35$.

(4) Write down the last term in the expansion in multiples of $\cos\theta$, of

(i) $2^{2n}\cos^{2n}\theta$. (ii) $2^{2n+1}\cos^{2n+1}\theta$. (iii) $2^{4n+2}\sin^{4n+2}\theta$.

(5) Any general formula expressed in cosines is also true in hyperbolic cosines.

(6) Any general formula expressed in cosines or in squares of sines will be true in hyperbolic cosines and sines if we write $-\sinh^2\theta$ for $\sin^2\theta$.

58. *To prove, when* n *is a positive integer, that cos* na *can be expressed in powers of cos* a*; that cosh* nx *can be expressed in powers of cosh* x*; and that the two expressions are the* **same** *in form.*

We have, when n is a positive integer

$$2\cos(n+1)a = 4\cos na \,.\, \cos a - 2\cos(n-1)a \qquad \text{(I.)}$$

$$2\cosh(n+1)x = 4\cosh nx \,.\, \cosh x - 2\cosh(n-1)x \qquad \text{(II.)}$$

In I. put $n=1$, and we obtain

$$2\cos 2a = 4\cos^2 a - 2.$$

Next put $n=2$, and using this last result, we have

$$2\cos 3a = 8\cos^3 a - 6\cos a$$

Put $n=3$, then using the last two results we have

$$2\cos 4a = 16\cos^4 a - 16\cos^2 a + 2$$

Next put $n=4$, then by the aid of the last two results, we can obtain an expression for $2\cos 5a$ in powers of $\cos a$, and so on.

By proceeding in this way we could obtain an expression in the required form for $2\cos na$ when n is any positive integer.

Again, by making use of (II.) in the same manner we could obtain an expression in the required form for $2\cosh nx$.

Also, since the *process is the same in each case*, the two resulting expressions are the *same* in form.

Example. Prove that cosh nx − *cos* na *is divisible by cosh* x − *cos* a *when* n *is a positive integer.*

From the above we have

$$\cosh nx = A_n \cosh^n x + A_{n-2}\cosh^{n-2} x + \text{etc.}$$
$$\cos na = A_n \cos^n a + A_{n-2}\cos^{n-2} a + \text{etc.}$$

the *coefficients* in the two expressions being the *same*.

Hence by subtraction

$$\cosh nx - \cos n\alpha = A_n(\cosh^n x - \cos^n\alpha) + A_{n-2}(\cosh^{n-2} x - \cos^{n-2}\alpha) + \ldots$$

and each term in this expression is divisible by $\cosh x - \cos\alpha$; [for $y^n - z^n$ is divisible by $y - z$ when n is a positive integer] therefore also $\cosh nx - \cos n\alpha$ is divisible by $\cosh x - \cos\alpha$

EXAMPLES. XXIII.

(1) Prove that

(i) $2\cos 6\theta = 64\cos^6\theta - 96\cos^4\theta + 36\cos^2\theta - 2$,

(ii) $e^{6x} + e^{-6x} = (e^x + e^{-x})^6 - 6(e^x + e^{-x})^4 + 9(e^x + e^{-x})^2 - 2$

(2) Divide $\cosh 6x - \cos 6\theta$ by $\cosh x - \cos\theta$.

(3) Prove that $x^n + x^{-n} - 2\cos n\alpha$
is divisible by $x + x^{-1} - 2\cos\alpha$ when n is a positive integer

(4) Prove that

$$2\cosh 5x = 32\cosh^5 x - 40\cosh^3 x + 10\cosh x$$

Prove that

(5) $\cosh 2x - \cos 2\theta = 2(\sin^2\theta\cosh^2 x + \cos^2\theta\sinh^2 x)$

59. We can find the law of the coefficients in the expansion for $2\cosh nx$, as follows: since

$$(1 - e^x z)(1 - e^{-x}z) = 1 - z(e^x + e^{-x}) + z^2$$
$$= 1 - z2\cosh x + z^2,$$

therefore $\log(1 - e^x z) + \log(1 - e^{-x}z)$

$$= \log\{1 - z(2\cosh x - z)\}.$$

Expanding each side of this identity by Art. 4, we have,

$$e^x z + \tfrac{1}{2}e^{2x}z^2 + \tfrac{1}{3}e^{3x}z^3 + \ldots + e^{-x}z + \tfrac{1}{2}e^{-2x}z^2 + \tfrac{1}{3}e^{-3x}z^3 + \ldots$$
$$= z(2\cosh x - z) + \tfrac{1}{2}z^2(2\cosh x - z)^2 + \tfrac{1}{3}z^3(2\cosh x - z)^3 + \ldots$$

This *identity* is true for all values of z which make the series convergent. Hence the coefficients of z^n on the two sides must be equal.

On the left-hand side the coefficient of z^n is

$$\frac{1}{n}(e^{nx}+e^{-nx}), \text{ that is } \frac{1}{n}2\cosh nx$$

On the right-hand side

the coefficient of z^n in $\frac{1}{n}z^n(2\cosh x - z)^n$ is $\frac{1}{n}2^n\cosh^n x$,

" " in $\frac{1}{n-1}z^{n-1}(2\cosh x - z)^{n-1}$ is

$$-\frac{1}{n-1}(n-1)2^{n-2}\cosh^{n-2}x,$$

and so on

Equating the coefficients we have

$$\frac{2}{n}\cosh nx = \frac{2^n}{n}\cosh^n x - \frac{1}{n-1}(n-1)2^{n-2}\cosh^{n-2}x$$
$$+\frac{1}{n-2}\frac{(n-2)(n-3)}{1\,.\,2}2^{n-4}\cosh^{n-4}x$$
$$-\frac{1}{n-3}\frac{(n-3)(n-4)(n-5)}{1\,.\,2\,.\,3}2^{n-6}\cosh^{n-6}x + \text{etc.}$$

Or, $2\cosh nx = 2^n\cosh^n x - n\,2^{n-2}\cosh^{n-2}x +$

$$n\frac{n-3}{1\,.\,2}2^{n-4}\cosh^{n-4}n - n\frac{(n-4)(n-5)}{1\,.\,2\,.\,3}2^{n-6}\cosh^{n-6}x + \,..$$

Therefore also $2\cos n\alpha = 2^n\cos^n\alpha - n2^{n-2}\cos^{n-2}\alpha +$

$$n\frac{(n-3)}{1\,.\,2}2^{n-4}\cos^{n-4}\alpha - n\frac{(n-4)(n-5)}{1\,.\,2\,.\,3}2^{n-6}\cos^{n-6}\alpha + \ldots$$

*60. This result may be transformed into a more symmetrical form as follows. The general term is

$$(-1)^r \frac{n(n-r-1)\ldots(n-2r+1)}{\underline{|r}}(2\cos\theta)^{n-2r}.$$

I. Let n be **even**, let $2m$ stand for n.

Then $n-2r$ is even; let $2p$ stand for $n-2r$.

The general term may be written

$$(-1)^{m-p}\frac{n(m+p-1)(m+p-2)\ldots(2p+1)}{\underline{|m-p}}2^{2p}\cos^{2p}\theta$$

[where p is to have all integral values from 0 to m]

$$=(-1)^{m-p}\frac{n\,\underline{|m+p-1}}{\underline{|m-p}\,\underline{|2p}}2^{2p}\cos^{2p}\theta$$

$$=(-1)^{m-p}\frac{n(m+p-1)(m+p-2)\ldots(m-p+1)}{\underline{|2p}}2^{2p}\cos^{2p}\theta$$

$$=(-1)^{m-p}\times$$

$$\frac{n(2m+2p-2)(2m+2p-4)\ldots(2m)\ldots(2m-2p+2)}{\underline{|2p}}2\cos^{2p}\theta$$

[for there are $2p-1$ terms in the series $(m+p-1)$, $(m+p-2)$ ...$(m-p+1)$]

$$=(-1)^{m-p}\frac{\{n^2-(2p-2)^2\}\{n^2-(2p-4)^2\}\ldots(n^2)}{\underline{|2p}}2\cos^{2p}\theta.$$

Hence putting for p the values 0, 1, 2... we have, $2\cos n\theta$

$$=(-1)^{\frac{n}{2}}2\left\{1-\frac{n^2}{1.2}\cos^2\theta+\frac{n^2(n^2-2^2)}{1.2.3.4}\cos^4\theta-\text{etc.}\right\}.$$

II. Let n be **odd**, then by putting $2m+1$ for n, and making a similar transformation, we shall obtain

$$2\cos n\theta=(-1)^{\frac{n-1}{2}}\times$$

$$2\left\{n\cos\theta-\frac{n(n^2-1^2)}{1.2.3}\cos^3\theta+\frac{n(n^2-1^2)(n^2-3^2)}{1.2.3.4.5}\cos^5\theta-\text{etc.}\right\}.$$

*61. The following is an illustration of an important method.

Suppose that we have a general theorem such as

$$\cos n\theta = A_0 + A_1 \cos\theta + A_2 \cos^2\theta + A_3 \cos^3\theta + \text{etc.}\ldots\ldots \quad \text{(I)},$$

which is true for all values of θ. For θ put $\theta + h$ and we have

$$\cos n\theta \,.\, \cos nh - \sin n\theta \,.\, \sin nh$$
$$= A_0 + A_1 (\cos\theta \,.\, \cos h - \sin\theta \,.\, \sin h)$$
$$+ A_2 (\cos\theta \,.\, \cos h - \sin\theta \,.\, \sin h)^2$$
$$+ A_3 (\cos\theta \,.\, \cos h - \sin\theta \,.\, \sin h)^3 + \text{etc.}$$

For $\cos h$ we may write $1 - Rh^2$ and for $\sin h$ we may write $h - R'h^3$, where R and R' are both finite when $h = 0$; hence we obtain [Art. 44.]

$$\cos n\theta - n^2h^2R\cos n\theta - nh\sin n\theta + n^3h^3R'\sin n\theta$$
$$= A_0 + A_1 \cos\theta + A_2 \cos^2\theta + \text{etc.}$$
$$- h\{A_1 \sin\theta + 2A_2 \sin\theta \,.\, \cos\theta$$
$$+ 3A_3 \sin\theta \,.\, \cos^2\theta + 4A_4 \sin\theta \,.\, \cos^3\theta + \ldots\}$$
$$+ \text{terms containing higher powers of } h \ \ldots\ldots\ldots \quad \text{(II)}.$$

This result is true for all values of h, and remembering I. we see that it is divisible by h. Dividing by h we get a result which is true for all values of h, and is therefore true in the limit when $h = 0$. Proceeding to the limit we obtain

$$+ n\sin n\theta = A_1 \sin\theta + 2A_2 \sin\theta\cos\theta + 3A_3 \sin\theta\cos^2\theta + \text{etc.}$$

*62. The student who is familiar with the methods of the Differential Calculus will observe that the above result may be obtained by differentiating each side of the equation (I).

*63. Applying this result to the series of Art. 60 we have, when n is **even**

$$2 \sin n\theta = (-1)^{\frac{n}{2}+1} 2 \sin \theta \times$$

$$\left\{ n \cos \theta - \frac{n(n^2-2^2)}{1.2.3} \cos^3 \theta + \frac{n(n^2-2^2)(n^2-4^2)}{1.2.3.4.5} \cos^5 \theta - \text{etc.} \right\},$$

when n is **odd**

$$2 \sin n\theta = (-1)^{\frac{n-1}{2}} 2 \sin \theta \times$$

$$\left\{ 1 - \frac{(n^2-1^2)}{1.2} \cos^2 \theta + \frac{(n^2-1^2)(n^2-3^2)}{1.2.3.4} \cos^4 \theta - \text{etc.} \right\}.$$

*64. Hence we have the series:

I. (n **even**), $(-1)^{\frac{n}{2}} \cos n\theta = 1 - \frac{n^2}{1.2} \cos^2 \theta + \frac{n^2(n^2-2^2)}{1.2.3.4} \cos^4 \theta -$

II. ,, $(-1)^{\frac{n}{2}+1} \sin n\theta = \sin \theta \left\{ n \cos \theta - \frac{n(n^2-2^2)}{1.2.3} \cos^3 \theta + \right\}$

III. (n **odd**), $(-1)^{\frac{n-1}{2}} \cos n\theta = n \cos \theta - \frac{n(n^2-1^2)}{1.2.3} \cos^3 \theta + \dots$

IV. ,, $(-1)^{\frac{n-1}{2}} \sin n\theta = \sin \theta \left\{ 1 - \frac{n^2-1^2}{1.2} \cos^2 \theta + \dots \right\}$.

65. In each of the above formulæ put $\frac{\pi}{2} - \theta$ for θ, then

I. (n **even**), $\cos n\theta = 1 - \frac{n^2}{1.2} \sin^2 \theta + \frac{n^2(n^2-2^2)}{1.2.3.4} \sin^4 \theta - \dots$

II. ,, $\sin n\theta = \cos \theta \left\{ n \sin \theta - \frac{n(n^2-2^2)}{1.2.3} \sin^3 \theta + \dots \right\}$

III. (n **odd**), $\sin n\theta = n \sin \theta - \frac{n(n^2-1^2)}{1.2.3} \sin^3 \theta + \dots$

IV. ,, $\cos n\theta = \cos \theta \left\{ 1 - \frac{n^2-1^2}{1.2} \sin^2 \theta + \dots \right\}$.

*66. In the following example an independent proof is given of the result of Art. 63.

Example. To expand $\cos n\theta$ *in ascending powers of* $\sin\theta$.

From Art. 19, we have when n is *even*

$$\cos n\theta = 1 + A_2\sin^2\theta + A_4\sin^4\theta + \text{etc.} \qquad \text{.I.}$$

The constant term is 1, because when $\theta=0$, $\cos n\theta=1$, and $\sin\theta=0$.

For θ write $\theta+h$ and we have

$$\cos n\theta\cos nh - \sin n\theta\sin nh$$
$$= 1 + A_2(\sin\theta\cos h + \cos\theta\sin h)^2 + A_4(\sin\theta\cos h + \cos\theta\sin h)^4 + \,..$$

For $\cos nh$, write $1-\frac{1}{2}n^2h^2+\text{etc.}$, for $\sin nh$, write $nh-\frac{1}{6}n^3h^3+\text{etc.}$, and substitute similar expressions for $\cos h$ and $\sin h$.

Then,

$$\cos n\theta\,\{1-\tfrac{1}{2}n^2h^2+\quad\} - \sin n\theta\,\{nh-\tfrac{1}{6}n^3h^3+\ldots\}$$
$$= 1 + A_2[\sin\theta\,(1-\tfrac{1}{2}h^2+\ldots)+\cos\theta\,(h-\tfrac{1}{6}h^3+\ldots)]^2$$
$$+ A_4[\sin\theta\,(1-\tfrac{1}{2}h^2+\,.\,.)+\cos\theta\,(h-\tfrac{1}{6}h^3+\ldots)]^4+\ldots$$

In this result, which is true for all values of h, we may equate the coefficient of h^2, and we have

$$-\tfrac{1}{2}n^2\cos n\theta = A_2[\cos^2\theta-\sin^2\theta]+$$

To obtain the general relation between the coefficients A_2, A_4,... consider the term

$$A_{2r}[\sin\theta\,(1-\tfrac{1}{2}h^2)+\cos\theta\,(h-\tfrac{1}{6}h^3\ldots)]^{2r},$$

or neglecting higher powers of h than h^2

$$A_{2r}[\sin\theta+h\cos\theta-\tfrac{1}{2}h^2\sin\theta]^{2r}$$

or expanding

$$A_{2r}\Big[\sin^{2r}\theta + 2r\sin^{2r-1}\theta\,(h\cos\theta-\tfrac{1}{2}h^2\sin\theta)$$
$$+\frac{2r(2r-1)}{1.2}\sin^{2r-2}\theta h^2\cos^2\theta\Big].$$

In this the coefficient of h^2 is

$$A_{2r}\left[-r\sin^{2r}\theta+\frac{2r(2r-1)}{1.2}\sin^{2r-2}\theta\cos^2\theta\right],$$

that is,

$$A_{2r}\left[-r\sin^{2r}\theta+\frac{2r(2r-1)}{1.2}\sin^{2r-2}\theta(1-\sin^2\theta)\right],$$

that is,

$$A_{2r}\left[\frac{2r(2r-1)}{1.2}\sin^{2r-2}\theta-\sin^{2r}\theta\left(r+\frac{2r(2r-1)}{1.2}\right)\right].$$

Hence we obtain as the coefficient of $\sin^{2r}\theta$ in the expansion of

$$-\tfrac{1}{2}n^2\cos n\theta$$

$$-A_{2r}\left\{r+\frac{2r(2r-1)}{1.2}\right\}+A_{2r+2}\left\{\frac{(2r+2)(2r+1)}{1.2}\right\}.$$

But A_{2r} is the coefficient of $\sin^{2r}\theta$ in the expansion of $\cos n\theta$; hence

$$-\tfrac{1}{2}n^2A_{2r}=-A_{2r}\left\{\frac{2r(2r-1)}{1.2}+r\right\}+A_{2r+2}\frac{(2r+2)(2r+1)}{1.2},$$

or
$$A_{2r+2}=-\frac{n^2-(2r)^2}{(2r+1)(2r+2)}A_{2r}.$$

Putting for r the values 1, 2, 3 . in succession we obtain

$$A_2=-\frac{n^2}{1.2}A_0=-\frac{n^2}{1.2},$$

$$A_4=-\frac{n^2-2^2}{3.4}A_2=\frac{n^2(n^2-2^2)}{1.2.3.4},$$

and so on.

Thus $\cos n\theta=1-\dfrac{n^2}{1.2}\sin^2\theta+\dfrac{n^2(n^2-2^2)}{1.2.3.4}\sin^4\theta-\text{etc.}$

[The same result would be obtained by comparing the series I. with that obtained by equating the *second differentials* of each side.]

*EXAMPLES. XXIV.

Prove the following statements:

(1) $\cos 4\theta=1-8\cos^2\theta+8\cos^4\theta$.

(2) $-\cos 6\theta=1-18\cos^2\theta+48\cos^4\theta-32\cos^6\theta$.

(3) $\cos 9\theta=9\cos\theta-120\cos^3\theta+432\cos^5\theta-576\cos^7\theta+2^9\cos^9\theta$.

(4) $\cos 6\theta=1-18\sin^2\theta+48\sin^4\theta-32\sin^6\theta$.

(5) $\cos(x+h)=\cos x - h\sin x - \frac{h^2}{\lfloor 2}\cos x + \frac{h^3}{\lfloor 3}\sin x + \text{etc.}$

(6) $\sin(x+h)=\sin x + h\cos x - \frac{h^2}{\lfloor 2}\sin x - \frac{h^3}{\lfloor 3}\cos x + \text{etc.}$

(7) $\cosh(x+h)=\cosh x + h\sinh x + \frac{h^2}{\lfloor 2}\cosh x + \text{etc.}$

(8) $\sinh(x+h)=\sinh x + h\cosh x + \frac{h^2}{\lfloor 2}\sinh x + \text{etc.}$

(9) $\log(x+h)=\log x + \frac{h}{x} - \frac{1}{2}\frac{h^2}{x^2} + \frac{1}{3}\frac{h^3}{x^3}$.

(10) If $\sin n\alpha = A_1\sin\alpha + A_3\sin^3\alpha + \ldots$, then

$$n\cos n\alpha = A_1\cos\alpha + 3A_3\cos\alpha\sin^2\alpha + \text{etc.}$$

(11) Prove that if $\cos^n\alpha = A_n\cos n\alpha + A_{n-2}\cos(n-2)\alpha + \text{etc.}$ then $n\sin\alpha\,.\cos^{n-1}\alpha = nA_n\sin n\alpha + (n-2)A_{n-2}\sin(n-2)\alpha + \text{etc.}$

(12) Prove that $2^n\sin\alpha\cos^{n-1}\alpha = 2\sin n\alpha + 2(n-2)\sin(n-2)\alpha$
$+ (n-1)(n-4)\cos(n-4)\alpha + \text{etc.}$

(13) Prove that

$$\sin n\alpha \div \sin\alpha = (2\cos\alpha)^{n-1} - (n-2)(2\cos\alpha)^{n-3} + (n-3)(n-4)(2\cos\alpha)^{n-5} - \text{etc.}$$

(14) Assuming when n is odd that

$$\sin n\theta = n\sin\theta + A_3\sin^3\theta + A_5\sin^5\theta + \ldots$$

prove as in the Example on page 68, that

$$\sin n\theta = n\sin\theta - \frac{n(n^2-1)}{1\,.\,2\,.\,3}\sin^3\theta + \frac{n(n^2-1)(n^2-3^2)}{1\,.\,2\,.\,3\,.\,4\,.\,5}\sin^5\theta \quad .$$

(15) If $\sinh nx = A_1\sinh x + A_3\sinh^3 x + \text{etc.}$, then

$$n\cosh nx = A_1\cosh x + 3A_3\cosh x\sinh^2 x +$$

*67. Consider the equation (n even)

$$2^n\cos^n\theta - n2^{n-2}\cos^{n-2}\theta + \frac{n(n-3)}{1\,.\,2}2^{n-4}\cos^{n-4}\theta + \text{etc.}$$

$$+(-1)^{\frac{n}{2}}\frac{n^2(n^2-2^2)}{\lfloor 4}2\cos^4\theta - (-1)^{\frac{n}{2}}\frac{n^2}{1\,.\,2}2\cos^2\theta + (-1)^{\frac{n}{2}}2$$

$$= 2\cos n\alpha.$$

This is an equation of the nth degree in $\cos\theta$, one of whose roots is $\cos a$; and since $\cos na = \cos n\left(a + \frac{2r\pi}{n}\right)$, the other roots are found by giving r the values 1, 2, 3 . $(n-1)$ in the expression $\cos\left(a + \frac{2r\pi}{n}\right)$. This result is useful in solving problems on symmetrical functions of the series

$$\cos a, \ \cos\left(a + \frac{2\pi}{n}\right), \quad \cos\left(a + \frac{4\pi}{n}\right), \text{ etc.}$$

Example. Find the sum of the series

$$\sec^2 a + \sec^2\left(a + \frac{2\pi}{n}\right) + \sec^2\left(a + \frac{4\pi}{n}\right) + \quad \text{to } n \text{ terms.}$$

That is, find the sum of the squares of the reciprocals of the roots of the above equation.

Hence, if S_0, S_1, S_2, etc. are the coefficients of $\cos^n\theta$, $\cos^{n-1}\theta$, etc. in the above equation, the required sum $= \left(\frac{S_{n-1}}{S_n}\right)^2 - \frac{2S_{n-2}}{S_n}$;

and when n is even

$$-S_n = \{2\cos na - (-1)^{\frac{n}{2}} 2\}, \qquad S_{n-1} = 0, \qquad S_{n-2} = -(-1)^{\frac{n}{2}} n^2.$$

The required sum is $\dfrac{n^2}{1-(-1)^{\frac{n}{2}}\cos na}$. When n is odd, the sum is $\dfrac{n^2}{\cos^2 na}$.

*EXAMPLES. XXV.

Find the value of the following expressions, in which $n\phi = 2\pi$.

(1) $\cos a \cos(a+\phi)\cos(a+2\phi)\ldots\cos\{a+(n-1)\phi\}$.

(2) $\sec a + \sec(a+\phi) + \sec(a+2\phi) + \ldots$ to n terms.

(3) $\sin a \sin(a+\phi)\sin(a+2\phi) + \ldots$ to n factors.

(4) $\operatorname{cosec}^2 a + \operatorname{cosec}^2(a+\phi) + \operatorname{cosec}^2(a+2\phi)\ldots$ to n terms.

(5) $\tan^2 a + \tan^2(a+\frac{1}{2}\phi) + \tan^2(a+\phi) +$ to n terms.

(6) $\tan a + \tan(a+\frac{1}{2}\phi) + \tan(a+\phi) +$ to n terms.

(7) $\cot a + \cot(a+\frac{1}{2}\phi) + \cot(a+\phi) + \ldots$ to n terms.

*68. *To find the Quadratic Factors of* $x^n - 2\cos n\alpha + x^{-n}$, when n is a whole number. [Compare Arts. 38, 58 *Ex.*]

The following is an identity:

$$x^n - 2\cos n\alpha + x^{-n} = \{x^{n-1} + x^{-(n-1)}\}\{x - 2\cos\alpha + x^{-1}\}$$
$$+ 2\cos\alpha\{x^{n-1} - 2\cos(n-1)\alpha + x^{-(n-1)}\}$$
$$- \{x^{n-2} - 2\cos(n-2)\alpha + x^{-(n-2)}\}.$$

Let $f(n)$ stand for $x^n - 2\cos n\alpha + x^{-n}$.

Then the above identity may be written

$$f(n) = \{x^{n-1} + x^{-(n-1)}\}f(1) + 2\cos\alpha f(n-1) - f(n-2).$$

Now from this it is clear that if $f(1)$ divides both $f(n-1)$ and $f(n-2)$, it must also divide $f(n)$.

But $f(1)$ does divide $f(1)$ and $f(2)$. Therefore $f(1)$ divides $f(3)$; and therefore $f(n)$, when n is any positive integer. That is, $(x - 2\cos\alpha + x^{-1})$ is a factor of $(x^n - 2\cos n\alpha + x^{-n})$.

Again, it follows that $x - 2\cos\left(\alpha + \frac{2r\pi}{n}\right) + x^{-1}$ is a factor of $x^n - 2\cos(n\alpha + 2r\pi) + x^{-n}$, that is of $x^n - 2\cos n\alpha + x^{-n}$.

Hence we get that the n factors of $x^n - 2\cos n\alpha + x^{-n}$ are

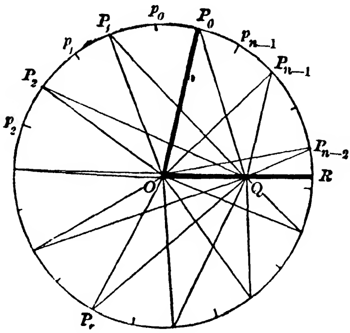
$$\{x - 2\cos\alpha + x^{-1}\}\left\{x - 2\cos\left(\alpha + \frac{2\pi}{n}\right) + x^{-1}\right\}\ldots\ldots$$

$$n \text{ factors up to } \left\{x - 2\cos\left(\alpha + \frac{2(n-1)\pi}{n}\right) + x^{-1}\right\}.$$

69. Writing $\frac{x}{a}$ for x in the result of Arts. 38, 58, or 68, and simplifying, we get

$$x^{2n} - a^n x^n \,.\, 2\cos n\alpha + a^{2n} = \{x^2 - ax \,.\, 2\cos\alpha + a^2\}$$
$$\times \left\{x^2 - ax \,.\, 2\cos\left(\alpha + \frac{2\pi}{n}\right) + a^2\right\} \times \text{etc. } n \text{ factors.}$$

70. This result may be interpreted geometrically.



Let OR be the initial line; with centre O and radius equal to x describe a circle, let ROP_0 be the angle α.

Divide the whole circumference, starting from P_0, into n equal parts, P_0P_1, P_1P_2, $..P_{n-1}P_0$.

Let OQ be equal to a, so that Q is any point in OR or in OR produced, then

$$QP_0^2 = OP_0^2 + OQ^2 - OP_0 . OQ . 2\cos ROP_0$$
$$= x^2 + a^2 - ax . 2\cos\alpha$$
$$QP_1^2 = OP_1^2 + OQ^2 - OP_1OQ . 2\cos ROP_1$$
$$= x^2 + a^2 - ax . 2\cos\left(\alpha + \frac{2\pi}{n}\right) \text{ and so on.}$$

Hence the result of Art. 68 may be written

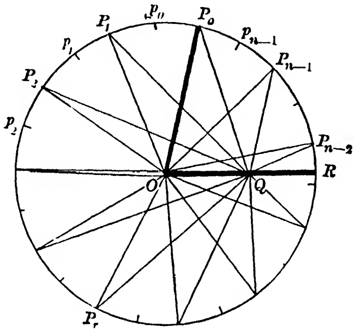
$$OP_0^{2n} - OP_0^n . OQ^n 2\cos nROP_0 + OQ^{2n} = QP_0^2 . QP_1^2 . QP_2^2 \ldots QP_{n-1}^2.$$

This result is **De Moivre's property of the circle.**

71. Some particular cases of the above should be noticed.

When Q coincides with R, $a = x$, and the above after taking the square root of both sides becomes

$$OR^n \cdot 2\sin(\tfrac{1}{2}n \,.\, ROP_0) = RP_0 \,.\, RP_1 \,.\, RP_2 \,.\,.\, RP_{n-1} \ldots \text{(I)}.$$



Again if R coincides with one of the points P, then a is a multiple of $\dfrac{2\pi}{n}$, and na is a multiple of 2π and we have

$$(OR^n - OQ^n)^2 = QP_0^2 \,.\, QP_1^2 \,.\, QP_2^2 \ldots QP^2_{n-1};$$

$$\therefore\ OR^n - OQ^n = QP_0 \,.\, QP_1 \,.\, QP_2 \quad .\, QP_{n-1} \ldots\ldots \text{(II)}.$$

Now if the arcs P_0P_1, $P_1P_2, \ldots$ are bisected in points p_0, p_1, $p_2 \ldots p_{n-1}$ respectively, we have by what has just been proved

$$OR^{2n} - OQ^{2n} = QP_0 \,.\, Qp_0 \,.\, QP_1 \,.\, Qp_1 \ldots QP_{n-1} \,.\, Qp_{n-1}.$$

Therefore, by division

$$OR^n + OQ^n = Qp_0 \,.\, Qp_1 \,.\, Qp_2 \ldots Qp_{n-1} \ldots\ldots \ldots \text{(III)}.$$

The student should notice carefully that in (I) Q lies somewhere *on* the circumference; in (II) OQ or OQ produced, passes through *one of the points* $P_0P_1 \ldots$; in (III) OQ or OQ produced, passes through *the middle point of one of the arcs* p_0p_1, p_1p_2, etc.

(II) and (III) are **Cotes' properties of the circle.**

MISCELLANEOUS EXAMPLES. XXVI.

(1) Prove Euler's Formula, viz.

$$\frac{\sin\theta}{\theta}=\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\dots\text{ ad inf.}$$

and deduce from it that $\sin\theta$ is greater than $\theta-\frac{1}{6}\theta^3$.

(2) AB is the diameter of a circle and Q_0 any point on the circumference; $Q_1, Q_2, Q_3, \dots$ are the points of bisection of the arcs $AQ_0, AQ_1, AQ_2, \dots$ prove that

$$BQ_1 . BQ_2 . BQ_3 \dots BQ_n = OA^n . \frac{AQ_0}{AQ_n}.$$

(3) Find the limit of $(\cos\theta)^{\cot\theta}$ when $\theta=0$.

(4) Find the limit of $\log_{\tan\theta}\sin\theta$ when $\theta=0$.

(5) Of what order is the error when $\dfrac{3\sin\theta}{2+\cos\theta}$ is substituted for θ?

(6) Prove that $2\cos n\phi-2\cos n\theta=2^n(\cos\phi-\cos\theta)\times$

$$\left\{\cos\phi-\cos\left(\frac{2\pi}{n}+\theta\right)\right\}\dots\left\{\cos\phi-\cos\left(\frac{\overline{2n-1}\pi}{n}+\theta\right)\right\}.$$

(7) Prove that

$$\tanh x+\tfrac{1}{3}\tanh^3x+\tfrac{1}{5}\tanh^5x+\dots=\tan x-\tfrac{1}{3}\tan^3x+\tfrac{1}{5}\tan^5x-\dots$$

(8) Expand $\cos^{4n}\theta+\sin^{4n}\theta$ in a series of cosines of multiples of θ.

(9) Prove that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}=\frac{1}{2^7}.$$

(10) Form the equation whose roots are

$$\tan^2\frac{\pi}{11},\ \tan^2\frac{2\pi}{11},\ \tan^2\frac{3\pi}{11},\ \tan^2\frac{4\pi}{11},\ \tan^2\frac{5\pi}{11}.$$

(11) $\log\dfrac{\tan\theta+i\tanh x}{\tan\theta-i\tanh x}=2i\tan^{-1}(\cot\theta\tanh x)$.

(12) $\dfrac{\sin 2\alpha\sin 4\alpha\dots\sin(2n-2)\alpha}{\sin\alpha\sin 3\alpha\sin 5\alpha\dots\sin(2n-1)\alpha}=n$,

where $2n\alpha=\pi$.

CHAPTER V.

ON THE SUMMATION OF TRIGONOMETRICAL SERIES.

72. There are two methods peculiarly applicable to the Summation of Trigonometrical Series.

FIRST METHOD.

73. Sometimes each term of a series may be transformed into the **difference** of two expressions.

Example i. *To sum the series*

$$\sin\alpha+\sin(\alpha+\delta)+\sin(\alpha+2\delta)+\ldots+\sin\{\alpha+(n-1)\delta\}$$

We have $2\sin\alpha\,.\,\sin\tfrac{1}{2}\delta$† $=\cos(\alpha-\tfrac{1}{2}\delta)-\cos(\alpha+\tfrac{1}{2}\delta)$,

$$2\sin(\alpha+\delta)\,.\,\sin\tfrac{1}{2}\delta=\cos(\alpha+\tfrac{1}{2}\delta)-\cos(\alpha+\tfrac{3}{2}\delta),$$

$$2\sin(\alpha+2\delta)\,.\,\sin\tfrac{1}{2}\delta=\cos(\alpha+\tfrac{3}{2}\delta)-\cos(\alpha+\tfrac{5}{2}\delta),$$

$$\ldots\ldots\ldots\ldots$$

$$2\sin\{\alpha+(n-1)\delta\}\,.\,\sin\tfrac{1}{2}\delta=\cos\left(\alpha+\frac{2n-3}{2}\delta\right)-\cos\left(\alpha+\frac{2n-1}{2}\delta\right).$$

Therefore, if S_n stands for the sum of n terms, we obtain by addition

$$2S_n\,.\,\sin\tfrac{1}{2}\delta=\cos(\alpha-\tfrac{1}{2}\delta)-\cos\left(\alpha+\frac{2n-1}{2}\delta\right)$$

$$=2\sin\{\alpha+\tfrac{1}{2}(n-1)\delta\}\,.\,\sin\tfrac{1}{2}n\delta.$$

Therefore $$S_n=\frac{\sin\{\alpha+\frac{1}{2}(n-1)\delta\}\,.\,\sin\frac{1}{2}n\delta}{\sin\frac{1}{2}\delta}.$$

Example ii. *To sum the series*

$$\cos\alpha+\cos(\alpha+\delta)+\cos(\alpha+2\delta)+\ldots\cos\{\alpha+(n-1)\delta\}.$$

We have $$2\cos\alpha\,.\,\sin\tfrac{1}{2}\delta=\sin(\alpha+\tfrac{1}{2}\delta)-\sin(\alpha-\tfrac{1}{2}\delta).$$

Hence, proceeding as in Example i., we obtain

$$S_n=\frac{\cos\{\alpha+\frac{1}{2}(n-1)\delta\}\,.\,\sin\frac{1}{2}n\delta}{\sin\frac{1}{2}\delta}.$$

† NOTE. The sum of a series of sines or cosines of angles in A. P. is found by multiplying each term by the sine of half the difference.

The results of these two examples are often useful. The student is advised to become familiar with them in *words.*

The sum of n *terms of a series of sines* (or cosines) *of angles in* A. P. *is equal to the sine* (or cosine) *of half the sum of the first and last angle, multiplied by the sine of* n *times half the difference, divided by the sine of half the difference.*

Example iii. *To prove that if* $n\phi = 2\pi$, *then*

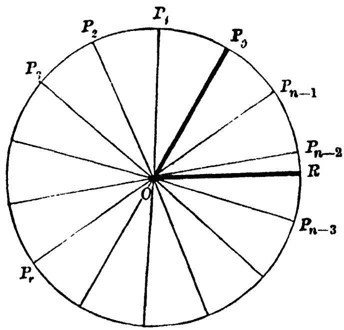
$$\sin \alpha + \sin (\alpha + \phi) + \sin (\alpha + 2\phi) \quad + \sin \{\alpha + (n-1)\phi\} = 0$$

for all values of α.

In the result of Example i., $\sin \frac{1}{2} n\delta$ occurs in the numerator, and $\sin \frac{1}{2} n\delta = \sin \frac{1}{2} n\phi = \sin \pi = 0$, and the denominator $\sin \frac{1}{2}\phi$ is *not* $=0$. Therefore the sum of the series $=0$. Similarly

$$\cos \alpha + \cos (\alpha + \phi) + \cos (\alpha + 2\phi) + \ldots + \cos \{\alpha + (n-1)\phi\} = 0$$

74. The results of Example iii. may be stated geometrically: Let OR be the initial line and ROP any angle, then if the whole circumference of a circle centre O and radius OR, be divided into n *equal parts* P_0P_1, P_1P_2, etc. Then the sum of the sines (or of the cosines) of all the angles ROP_0, $ROP_1 \ldots ROP_{n-1}$ is zero†.



† This is an expression of the fact that the centre of gravity of equal particles placed at the points $P_0P_1 \ldots$ is at the centre of the circle.

**Example* iv. To sum the series

$$\sin^m \alpha + \sin^m (\alpha + \delta) + \sin^m (\alpha + 2\delta) + \ldots + \sin^m \{\alpha + (n-1)\delta\}.$$

This may be done by the aid of Arts. 37, 55, 56.

Thus, if m be even

$$2^m \sin^m \alpha = (-1)^{\frac{m}{2}} 2 \{\cos m\alpha - m \cos (m-2)\alpha + \quad \}$$

and the required sum may be obtained from the known sum of the series $\{\cos m\alpha + \cos m(\alpha + \delta) + \cos m(\alpha + 2\delta) + \text{etc.}\}$

and $\{\cos (m-2)\alpha + \cos (m-2)(\alpha + \delta) + \cos (m-2)(\alpha + 2\delta) + \quad \}$ etc.

We may find the sum of the series

$$\cos^m \alpha + \cos^m (\alpha + \delta) + \cos^m (\alpha + 2\delta) + \text{etc. to } n \text{ terms}$$

in a similar manner by the aid of Art. 53.

Example v. *Sum* to n terms *the series*

$$\cos^3 \alpha + \cos^3 2\alpha + \cos^3 3\alpha +$$

$$8 \cos^3 \alpha = 2 \cos 3\alpha + 6 \cos \alpha,$$

we have therefore the required sum to n terms

$$= \tfrac{1}{8}[2 \cos 3\alpha + 2 \cos 6\alpha + 2 \cos 9\alpha +$$

$$+ 6 \cos \alpha + 6 \cos 2\alpha + 6 \cos 3\alpha + \ldots]$$

$$= \frac{2 \cos \{3\alpha + \frac{1}{2}(n-1)3\alpha\} \sin \frac{3}{2}n\alpha}{8 \sin \frac{3}{2}\alpha}$$

$$+ \frac{6 \cos \{\alpha + \frac{1}{2}(n-1)\alpha\} \sin \frac{1}{2}n\alpha}{8 \sin \frac{1}{2}\alpha},$$

by Example ii, p. 76.

EXAMPLES. XXVII.

Sum the following series to n terms.

(1) $\sin \alpha + \sin 2\alpha + \sin 3\alpha + \ldots\ldots$

(2) $\cos \alpha + \cos 3\alpha + \cos 5\alpha + \ldots\ldots$

(3) $\sin \alpha + \sin 4\alpha + \sin 7\alpha + \ldots\ldots$

(4) $\sin\alpha \,.\, \cos\alpha + \sin 2\alpha \,.\, \cos 2\alpha + \sin 3\alpha \,.\, \cos 3\alpha + \,..$

(5) $\cos^2\alpha + \cos^2 2\alpha + \cos^2 3\alpha +$

(6) $\sin^3\alpha + \sin^3 2\alpha + \sin^3 3\alpha +$

(7) $\cos^4\alpha + \cos^4 2\alpha + \cos^4 3\alpha + .$

(8) $\sin 2\alpha \,.\, \cos\alpha + \sin 3\alpha \,.\, \cos 2\alpha + \sin 4\alpha \,.\, \cos 3\alpha +$

(9) $\sin\alpha \,.\, \sin 2\alpha + \sin 2\alpha \,.\, \sin 3\alpha + \sin 3\alpha \,.\, \sin 4\alpha +$

(10) $\cos^3\alpha + \cos^3(\alpha+\delta) + \cos^3(\alpha+2\delta) +$

(11) $\sin^4\alpha + \sin^4(\alpha+\delta) + \sin^4(\alpha+2\delta) +$

(12) Solve the equation

$\sin\theta + \sin 2\theta + \sin 3\theta +$ etc. to n terms $= \cos\theta + \cos 2\theta + \cos 3\theta +$ etc. to n terms.

(13) Write down the value of series (10) and (11) when $n\delta = 2\pi$.

(14) Prove that

$$\frac{\sin\alpha + \sin 3\alpha + \sin 5\alpha + \ .\text{ to } n \text{ terms}}{\cos\alpha + \cos 3\alpha + \cos 5\alpha + \ .\text{ to } n \text{ terms}} = \tan n\alpha.$$

(15) Prove that

$$\frac{\sin\alpha + \sin(\alpha+\delta) + \sin(\alpha+2\delta) + \quad \text{to } (2n-1) \text{ terms}}{\sin\alpha + \sin(\alpha+2\delta) + \sin(\alpha+4\delta) \text{ to } n \text{ terms}}$$

is independent of α.

(16) Deduce from Ex. (1) the sum of the series

$$1 + 2 + 3 + \ldots\ldots + n.$$

(17) Deduce from Ex. (6) the sum of the series

$$1^3 + 2^3 + 3^3 + \text{etc.} + n^3.$$

(18) Deduce from Ex. (9) the sum of the series

$$1 \,.\, 2 + 2 \,.\, 3 + 3 \,.\, 4 + \text{etc.} + n(n+1).$$

(19) Sum the series $\sin\alpha - \sin(\alpha+\delta) + \sin(\alpha+2\delta) -$ etc. to n terms.

(20) Sum the series $\cos\alpha - \cos(\alpha+\delta) + \cos(\alpha+2\delta) -$ etc. to n terms.

(21) Prove that the series $\sin^m\alpha + \sin^m(\alpha+\phi) +$ etc. to n terms, where $n\phi = 2\pi$, is independent of α, provided m is less than n.

(22) Prove that the series $\cos^m\alpha + \cos^m(\alpha+\phi) +$ etc. to n terms, where $n\phi = 2\pi$, is independent of α if m is less than n.

Example v. *To sum* cosec θ + cosec 2θ + cosec 4θ + . *to* n *terms.*

We have
$$\operatorname{cosec}\theta = \cot\tfrac{1}{2}\theta - \cot\theta,$$
$$\operatorname{cosec}2\theta = \cot\theta - \cot 2\theta,$$
$$\dots$$
$$\operatorname{cosec}2^{n-1}\theta = \cot 2^{n-2}\theta - \cot 2^{n}\theta.$$

Therefore, as in Art. 73,
$$S_n = \cot\tfrac{1}{2}\theta - \cot 2^{n-1}\theta.$$

Example vi. *To sum* $\tan\theta + \frac{1}{2}\tan\frac{1}{2}\theta + \frac{1}{4}\tan\frac{1}{4}\theta$ + etc. *to* n *terms.*

We have
$$\tan\theta = \cot\theta - 2\cot 2\theta,$$
$$\tfrac{1}{2}\tan\tfrac{1}{2}\theta = \tfrac{1}{2}\cot\tfrac{1}{2}\theta - \cot\theta,$$
$$\tfrac{1}{4}\tan\tfrac{1}{4}\theta = \tfrac{1}{4}\cot\tfrac{1}{4}\theta - \tfrac{1}{2}\cot\tfrac{1}{2}\theta, \text{ etc.}$$

Therefore
$$S_n = \frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}} - 2\cot 2^{n-1}\theta.$$

75. If the result of summation of such series is given, it is often easy from that result to discover the required transformation.

For example. The result of the summation in Example i. has $\sin\frac{1}{2}\delta$ in the denominator. This suggests that $\sin\alpha \,.\, \sin\frac{1}{2}\delta$ may be transformed into two quantities which are of course $\frac{1}{2}\cos(\alpha - \frac{1}{2}\delta) - \frac{1}{2}\cos(\alpha + \frac{1}{2}\delta)$. Again, in Examples v. and vi. the required transformation will be at once seen if we *put* n = 1 *in the answer.*

The student however is advised only to resort to this method of solution as a last resource.

EXAMPLES. XXVIII.

Sum the following series to n terms.

(1) $\sec\theta \,.\, \sec 2\theta + \sec 2\theta \,.\, \sec 3\theta + \sec 3\theta \,.\, \sec 4\theta + \dots\dots$

(2) $\operatorname{cosec}\theta \,.\, \operatorname{cosec}2\theta + \operatorname{cosec}2\theta \,.\, \operatorname{cosec}3\theta + \operatorname{cosec}3\theta \,.\, \operatorname{cosec}4\theta + \dots\dots$

(3) $\operatorname{cosec}\theta \,.\, \sec 2\theta - \sec 2\theta \,.\, \operatorname{cosec}3\theta + \operatorname{cosec}3\theta \,.\, \sec 4\theta$ − etc.

(4) $\dfrac{1}{\cos\alpha+\cos 3\alpha}+\dfrac{1}{\cos\alpha+\cos 5\alpha}+\dfrac{1}{\cos\alpha+\cos 7\alpha}+\text{etc.}$

(5) $\dfrac{\sin\alpha}{\cos\alpha+\cos 2\alpha}+\dfrac{\sin 2\alpha}{\cos\alpha+\cos 4\alpha}+\dfrac{\sin 3\alpha}{\cos\alpha+\cos 6\alpha}+\text{etc.}$

(6) $\dfrac{\cos\alpha}{\cos\alpha-\cos 2\alpha}+\dfrac{\cos 2\alpha}{\cos\alpha-\cos 4\alpha}+\dfrac{\cos 3\alpha}{\cos\alpha-\cos 6\alpha}+\text{etc.}$

(7) $\sin 2\theta\,.\,\sin^2\theta+\frac{1}{2}\sin 4\theta\,.\,\sin^2 2\theta+\frac{1}{4}\sin 8\theta\,.\,\sin^2 4\theta+$

(8) $\sin 2\theta\,.\,\cos^2\theta-\frac{1}{2}\sin 4\theta\,.\,\cos^2 2\theta+\frac{1}{4}\sin 8\theta\,.\,\cos^2 4\theta-$

(9) $\sin\theta\,.\,\cos^2\dfrac{\theta}{2}-2\sin\dfrac{\theta}{2}\cos^2\dfrac{\theta}{4}+4\sin\dfrac{\theta}{4}\,.\,\cos^2\dfrac{\theta}{8}-\quad\ldots$

(10) $\sec\alpha\,.\,\sin 2\alpha\,.\,\sec 3\alpha+\sec 3\alpha\,.\,\sin 4\alpha\,.\,\sec 5\alpha+\text{etc.}$

(11) $\dfrac{\sin 2\alpha}{\sin\alpha\,.\,\sin 3\alpha}-\dfrac{\sin 4\alpha}{\sin 3\alpha\,.\,\sin 5\alpha}+\text{etc.}$

(12) $\tan^{-1}\dfrac{1}{1+1\,.\,2}+\tan^{-1}\dfrac{1}{1+2\,.\,3}+\tan^{-1}\dfrac{1}{1+3\,.\,4}+\text{etc.}$

(13) $\tan^{-1}\dfrac{\alpha}{1+1\,.\,2\alpha^2}+\tan^{-1}\dfrac{\alpha}{1+2\,.\,3\alpha^2}+\tan^{-1}\dfrac{\alpha}{1+3\,.\,4\alpha^2}+$

(14) $\tan^{-1}\dfrac{3\alpha^2}{1+1^2\,.\,2^2\alpha^4}+\tan^{-1}\dfrac{5\alpha^2}{1+2^2\,.\,3^2\alpha^4}+\text{etc}$

(15) $\tan\theta+2\tan 2\theta+4\tan 4\theta+\text{etc.}$

(16) $\tan\alpha+\cot\alpha+\tan 2\alpha+\cot 2\alpha+\tan 4\alpha+\cot 4\alpha+\text{etc.}$

(17) $\sin\theta\sin 3\theta+\sin 2\theta\sin 6\theta+\sin 2^2\theta\sin(2^2\,.\,3\theta)+\text{etc.}$

(18) $\sin\theta\,.\,\sin 3\theta+\sin\dfrac{\theta}{2}\sin\dfrac{3\theta}{2}+\sin\dfrac{\theta}{2^2}\,.\,\sin\dfrac{3\theta}{2^2}+$

(19) $\cot\theta\operatorname{cosec}\theta+2\cot 2\theta\operatorname{cosec} 2\theta+2^2\cot 2^2\theta\operatorname{cosec} 2^2\theta+\ldots$

(20) $\cot\theta\operatorname{cosec}\theta+\frac{1}{2}\cot\dfrac{\theta}{2}\operatorname{cosec}\dfrac{\theta}{2}+\dfrac{1}{2^2}\cot\dfrac{\theta}{2^2}\operatorname{cosec}\dfrac{\theta}{2^2}+$

(21) $\operatorname{cosec}\alpha+\operatorname{cosec}\dfrac{\alpha}{2}+\operatorname{cosec}\dfrac{\alpha}{2^2}+\ldots$

(22) $\dfrac{1}{2}\sec\alpha+\dfrac{1}{2^2}\sec\alpha\sec 2\alpha+\dfrac{1}{2^3}\sec\alpha\sec 2\alpha\sec 2^2\alpha+\text{etc.}$

(23) Deduce from Ex. 2 the sum of the series $\dfrac{1}{1\,.\,2}+\dfrac{1}{2\,.\,3}+\dfrac{1}{3\,.\,4}$ to n terms.

(24) Deduce from Ex. 17 the sum to n terms of

$$1\,.\,3+2^2\,.\,1\,.\,3+2^4\,.\,1\,.\,3+\ldots$$

SECOND METHOD.

76. When the sum of a series of the form

$$A_0 - A_1 x + A_2 x^2 + A_3 x^3 + \text{etc.}$$

is known, we can obtain the sums of two series of the forms

$$A_0 + A_1 x \cos\theta + A_2 x^2 \cos 2\theta + A_3 x^3 \cos 3\theta + \text{etc.},$$

and $$A_1 x \sin\theta + A_2 x^2 \sin 2\theta + A_3 x^3 \sin 3\theta + \text{etc.}$$

Let C stand for the sum of the first series, and S for the sum of the second series, then

$$C + iS = A_0 + A_1 x(\cos\theta + i\sin\theta) + A_2 x^2(\cos 2\theta + i\sin 2\theta) + \text{etc.}$$

$$= A_0 + A_1 x e^{i\theta} + A_2 (xe^{i\theta})^2 + A_3 (xe^{i\theta})^3 + \text{etc.}$$

The sum of the last series is known by hypothesis.

The result of the summation must then be expressed in the form $A + iB$; whence we have $C = A$ and $S = B$. [Art. 11.]

Example 1. *Sum the series*

$$1 + x\cos\theta + x^2\cos 2\theta + \text{etc.} + x^{n-1}\cos(n-1)\theta$$

Let C_n be the sum of this series, and let

$$S_{n-1} = x\sin\theta + x^2\sin 2\theta + \quad + x^{n-1}\sin(n-1)\theta$$

Then $C_n + iS_{n-1} = 1 + xe^{i\theta} + x^2e^{2i\theta} + x^3e^{3i\theta} + \quad x^{n-1}e^{(n-1)i\theta}$.

This is a series in G. P. $\therefore$ its sum $= \dfrac{1 - x^n e^{in\theta}}{1 - xe^{i\theta}}$.

Multiply the numerator and the denominator of this result each by $1 - xe^{-i\theta}$, and we have

$$C_n + iS_{n-1} = \frac{(1 - x^n e^{in\theta})(1 - xe^{-i\theta})}{1 - x(e^{i\theta} + e^{-i\theta}) + x^2} = \frac{1 - x^n e^{in\theta} - xe^{-i\theta} + x^{n+1}e^{i(n-1)\theta}}{1 - 2x\cos\theta + x^2}.$$

Therefore $C_n = \dfrac{1 - x\cos\theta - x^n\cos n\theta + x^{n+1}\cos(n-1)\theta}{1 - 2x\cos\theta + x^2}$. [Art. 11.]

Also $$S_{n-1} = \frac{x\sin\theta - x^n\sin n\theta + x^{n+1}\sin(n-1)\theta}{1 - 2x\cos\theta + x^2}.$$

Example 2. *Sum the infinite series*

$$\sin\alpha + x\sin(\alpha+\beta) + \frac{x^2}{\lfloor 2}\sin(\alpha+2\beta) + \frac{x^3}{\lfloor 3}\sin(\alpha+3\beta) + \dots$$

Let S stand for the sum of the series, and let

$$C = \cos\alpha + x\cos(\alpha+\beta) + \frac{x^2}{\lfloor 2}\cos(\alpha+2\beta) + \dots$$

Then $\quad C + iS = e^{i\alpha} + xe^{i\alpha}e^{i\beta} + \frac{x^2}{\lfloor 2}e^{i\alpha}e^{i2\beta} + \quad = e^{i\alpha}\{e^{xe^{i\beta}}\}$ [Art. 3]

$$= e^{i\alpha}e^{x(\cos\beta + i\sin\beta)}$$

$$= (\cos\alpha + i\sin\alpha)\, e^{x\cos\beta}\{\cos(x\sin\beta) + i\sin(x\sin\beta)\}.$$

Therefore $\quad S = e^{x\cos\beta}\{\sin\alpha\cos(x\sin\beta) + \cos\alpha\sin(x\sin\beta)\}$

$$= e^{x\cos\beta}\sin(\alpha + x\sin\beta).$$

EXAMPLES. XXIX.

Sum the following series to infinity.

(1) $\sin\alpha + x\sin 2\alpha + x^2\sin 3\alpha + \quad .$

(2) $\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta) + \quad .$

(3) $\sin\alpha + \cos\alpha\sin(\alpha+\beta) + \cos^2\alpha\sin(\alpha+2\beta) + \quad .$

(4) $\cos\alpha + \sin\alpha\cos(\alpha+\beta) + \sin^2\alpha\cos(\alpha+2\beta) + \quad .$

(5) $\sin\alpha + \frac{1}{\lfloor 2}\sin 2\alpha + \frac{1}{\lfloor 3}\sin 3\alpha + .$

(6) $x\sin\alpha + \frac{x^2}{\lfloor 2}\sin 2\alpha + \frac{x^3}{\lfloor 3}\sin 3\alpha +$

(7) $1 - \frac{\cos\alpha}{\lfloor 1}\cos\beta + \frac{\cos^2\alpha}{\lfloor 2}\cos 2\beta - \frac{\cos^3\alpha}{\lfloor 3}\cos 3\beta + \; ..$

(8) $\sin\alpha - \frac{1}{\lfloor 1}\cos\alpha\sin(\alpha+\beta) + \frac{1}{\lfloor 2}\cos^2\alpha\sin(\alpha+2\beta) - \quad .$

(9) $\sin\alpha - \frac{1}{2}\sin 2\alpha + \frac{1}{3}\sin 3\alpha - \; ..$

(10) $\cos\alpha + \frac{1}{2}\cos 2\alpha + \frac{1}{3}\cos 3\alpha + \quad .$

(11) $\sin\alpha \,.\, \cos\beta - \frac{1}{2}\sin^2\alpha \,.\, \cos 2\beta + \frac{1}{3}\sin^3\alpha \,.\, \cos 3\beta - \,.\,.$

(12) $\cos\theta \,.\, \cos\phi + \frac{1}{2}\cos^2\theta \,.\, \cos 2\phi + \frac{1}{3}\cos^3\theta \,.\, \cos 3\phi + \dots$

Sum the series 13, 14, 17 to 20, to n terms.

(13) $\cos\alpha + x\cos(\alpha+\beta) + x^2\cos(\alpha+2\beta) + \ldots$

(14) $x\sin\alpha - x^2\sin(\alpha+\beta) + x^3\sin(\alpha+2\beta) - \ldots$

(15) $1 + n\cos\alpha + \frac{n(n-1)}{\lfloor 2}\cos 2\alpha + \frac{n(n-1)(n-2)}{\lfloor 3}\cos 3\alpha +$ to $(n+1)$ terms.

(16) $\sin\alpha + nx\sin(\alpha+\beta) + \frac{n(n-1)}{\lfloor 2}x^2\sin(\alpha+2\beta) +$ to $(n+1)$ terms.

(17) $1 + \cos\alpha \,.\, \cos\beta + \cos^2\alpha \,.\, \cos 2\beta + \cos^3\alpha \,.\, \cos 3\beta + \ldots$

(18) $\sin\alpha + \sin\alpha \,.\, \sin(\alpha+\beta) + \sin^2\alpha \,.\, \sin(\alpha+2\beta) + \ldots$

(19) $\sin\alpha + 2\sin 2\alpha + 3\sin 3\alpha +$

(20) $1^2\cos\alpha + 2^2\cos 2\alpha + 3^2\cos 3\alpha +$

*Sum the following series to infinity.

(21) $\frac{p}{q}\cos\alpha\sin\beta + \frac{p(p-q)}{\lfloor 2\, q^2}\cos^2\alpha\sin 2\beta + \frac{p(p-q)(p-2q)}{\lfloor 3\, q^3}\cos^3\alpha\sin 3\beta +$

(22) $\cos\alpha - \frac{1}{3}\cos 3\alpha + \frac{1}{5}\cos 5\alpha - \ldots$

(23) $1 + \frac{1}{\lfloor 1}e^{\cos\alpha}\cos(\sin\alpha) + \frac{1}{\lfloor 2}e^{2\cos\alpha}\cos(2\sin\alpha) +$

(24) $e^{-x}\cos y - \frac{1}{3}e^{-3x}\cos 3y + \frac{1}{5}e^{-5x}\cos 5y -$

(25) $e^{x}\sin x - \frac{1}{2}e^{2x}\sin 2x + \frac{1}{3}e^{3x}\sin 3x -$

Expansion in Series.

77. The expression $x^2 - 2x\cos\theta + 1$ is the product of the two factors $(1 - xe^{\iota\theta})(1 - xe^{-\iota\theta})$, and therefore an expression having $x^2 - 2x\cos\theta + 1$ for its denominator may often be expanded in ascending powers of x by finding its equivalent partial fractions.

Example i. *Expand* $\dfrac{2\cos\alpha - 2x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2}$ *in ascending powers of* x.

$$\frac{2\cos\alpha - 2x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2} = \frac{e^{i\alpha}+e^{-i\alpha} - x(e^{i\alpha-i\beta}+e^{-i\alpha+i\beta})}{(1-xe^{i\beta})(1-xe^{-i\beta})} \quad \text{[Art. 23.]}$$

$$= \frac{e^{i\alpha}}{1-xe^{i\beta}} + \frac{e^{-i\alpha}}{1-xe^{-i\beta}}$$

$$= e^{i\alpha}\{1+xe^{i\beta}+x^2e^{2i\beta}+ \quad\} + e^{-i\alpha}\{1+xe^{-i\beta}+x^2e^{-2i\beta}+ \quad\}$$

$$= e^{i\alpha}+e^{-i\alpha}+x(e^{i\alpha+i\beta}+e^{-i\alpha-i\beta})+x^2(e^{i\alpha+2i\beta}+e^{-i\alpha-2i\beta})+$$

$$= 2\cos\alpha + x\,2\cos(\alpha+\beta)+x^2\,2\cos(\alpha+2\beta)+ \quad \text{[Art. 23.]}$$

Example ii. *Expand* $\dfrac{1-x^2}{1-2x\cos\alpha+x^2}$ *in ascending powers of* x.

Here, $(1-2x\cos\alpha+x^2)^{-1}$ may be written $(1-xe^{i\alpha})^{-1}(1-xe^{-i\alpha})^{-1}$

$$= \{1+xe^{i\alpha}+x^2e^{2i\alpha}+x^3e^{3i\alpha}+ \quad\}\{1+xe^{-i\alpha}+x^2e^{-2i\alpha}+x^3e^{-3i\alpha}+ \quad\}$$

$$= 1+x^2+x^4+ \ldots + (e^{i\alpha}+e^{-i\alpha})\{x+x^3+x^5+ \ldots\}$$

$$+(e^{2i\alpha}+e^{-2i\alpha})\{x^2+x^4+x^6+ \quad\}+\text{etc.}$$

$$= \frac{1}{1-x^2}\{1+x\,2\cos\alpha+x^2\,2\cos 2\alpha+x^3\,2\cos 3\alpha+ \quad\}$$

This gives the required expansion.

Writing $\dfrac{a}{b}$ for x we have, if $a>b$

$$(a^2-2ab\cos\alpha+b^2)^{-1} = \frac{1}{a^2-b^2}\left\{1+2\frac{a}{b}\cos\alpha + 2\frac{a^2}{b^2}\cos 2\alpha + \ldots\right\}.$$

Example iii. *In any triangle* $c^2 = a^2 - 2ab\cos C + b^2$; *let* a *be* $>$ b, *then*

$$c^2 = a^2\left(1-\frac{b}{a}e^{iC}\right)\left(1-\frac{b}{a}e^{-iC}\right), \quad \text{[Art. 77.]}$$

$$\therefore\ 2\log c = 2\log a + \log\left(1-\frac{b}{a}e^{iC}\right) + \log\left(1-\frac{b}{a}e^{-iC}\right)$$

$$= 2\log a - \frac{b}{a}(e^{iC}+e^{-iC}) - \tfrac{1}{2}\frac{b^2}{a^2}(e^{2iC}+e^{-2iC}) - \text{etc.} \quad \text{[Art. 4.]}$$

$$\log c = \log a - \frac{b}{a}\cos C - \tfrac{1}{2}\frac{b^2}{a^2}\cos 2C - \tfrac{1}{3}\frac{b^3}{a^3}\cos 3C - \text{etc.} \quad \text{[Art. 23.]}$$

This series may sometimes be made useful when $\dfrac{b}{a}$ is small.

Example iv. *Expand* $e^{ax} \sin bx$ *in ascending powers of* x.

$$2i \cdot e^{ax} \sin bx = e^{ax}(e^{ibx} - e^{-ibx})$$
$$= e^{x(a+ib)} - e^{x(a-ib)}.$$

Expanding $e^{x(a+ib)}$ and $e^{x(a-ib)}$ in ascending powers of x by Art. 2, we have as the coefficient of $\frac{x^n}{\lfloor n}$ in the expansion of $e^{ax} \sin bx$

$$\frac{1}{2i}\{(a+ib)^n - (a-ib)^n\}.$$

This is equal to

$$\frac{1}{2i}\{r^n(\cos\theta + i\sin\theta)^n - r^n(\cos\theta - i\sin\theta)^n\},$$

where $$r = \sqrt{(a^2+b^2)}, \quad \theta = \tan^{-1}\frac{b}{a},$$

that is, is equal to

$$\frac{1}{2i} r^n \{e^{in\theta} - e^{-in\theta}\},$$

or $$r^n \sin n\theta,$$

that is $$(a^2+b^2)^{\frac{n}{2}} \sin\left\{n \tan^{-1}\frac{b}{a}\right\}.$$

78. The following example is important.

Given $\sin\theta = x\sin(\theta+\alpha)$, *expand* θ *in a series of ascending powers of* x.

Since $\sin\theta = x\sin(\theta+\alpha)$, $\therefore e^{i\theta} - e^{-i\theta} = x(e^{i\theta+i\alpha} - e^{-i\theta-i\alpha})$, [Art. 23.]

$$\therefore \quad e^{2i\theta} - 1 = x\{e^{i\alpha}e^{2i\theta} - e^{-i\alpha}\}.$$

$$\therefore \quad e^{2i\theta} = \frac{1 - xe^{-i\alpha}}{1 - xe^{i\alpha}}.$$

$$\therefore \quad \log e^{2i\theta} = \log(1 - xe^{-i\alpha}) - \log(1 - xe^{i\alpha}).$$

$$\therefore \quad 2i\theta + 2ir\pi = x(e^{i\alpha} - e^{-i\alpha}) + \tfrac{1}{2}x^2(e^{2i\alpha} - e^{-2i\alpha}) + \text{etc.} \quad \text{[Art. 4.]}$$

$$\therefore \quad \theta + r\pi = x\sin\alpha + \tfrac{1}{2}x^2\sin 2\alpha + \tfrac{1}{3}x^3\sin 3\alpha + \dots \quad \text{[Art. 23, 28.]}$$

If in the above $x = -1$, then $\sin\theta = -\sin(\theta+\alpha)$, so that we may put -2θ for α. Hence we obtain when θ is less than $\frac{1}{2}\pi$

$$\theta = \sin 2\theta - \tfrac{1}{2}\sin 4\theta + \tfrac{1}{3}\sin 6\theta - \text{etc.}$$

EXAMPLES. XXX.

(1) Expand $\dfrac{1}{1-2a\cos\phi+a^2}$ in a series of ascending powers of a; and prove that if p_{n-1}, p_n, p_{n+1} be the coefficients of three consecutive terms $2p_n\cos\phi=p_{n-1}+p_{n+1}$.

Expand the following expressions in ascending powers of x

(2) $\dfrac{\sin\alpha}{1-2x\cos\alpha+x^2}$.

(3) $\dfrac{1+x\cos\theta}{1+2x\cos\theta+x^2}$.

(4) $\log(1-2x\cos\alpha+x^2)$.

(5) $\dfrac{\sin\alpha-x\sin(\alpha-\beta)}{1-2x\cos\beta+x^2}$.

(6) $e^{x\cos\alpha}\cos(x\sin\alpha)$.

(7) $e^{ax}\cos bx$.

(8) $e^{ax}\cos bx+e^{bx}\cos ax$.

(9) $e^{x\cos\alpha}\sin(x\sin\alpha)$.

(10) $e^{x\cos\beta}\cos(\alpha+x\sin\beta)$.

(11) $e^{x\cos\beta}\sin(\alpha+x\sin\beta)$.

(12) In any triangle $\sin A=\dfrac{a}{b}\sin(A+C)$, hence prove that

$$A=\frac{a}{b}\sin C+\frac{a^2}{2b^2}\sin 2C+\text{etc}$$

(13) If $\tan\phi=n\tan\theta$, find a series for ϕ in terms of θ.

(14) Prove that $\dfrac{\sec^n\theta}{2^n}=\frac{1}{2}\left\{\dfrac{e^{in\theta}}{(1+e^{2i\theta})^n}+\dfrac{e^{-in\theta}}{(1+e^{-2i\theta})^n}\right\}$, and expand $\sec^n\theta$ in cosines of multiples of θ.

(15) Prove that $\cos n\alpha\cos^n\alpha+i\sin n\alpha\cos^n\alpha=\dfrac{1}{(1-i\tan\alpha)^n}$ and expand $\cos n\alpha\cos^n\alpha$ in ascending powers of $\tan\alpha$.

(16) Sum to infinity the series

(i) $4+9\cos\theta+21\cos 2\theta+51\cos 3\theta+\text{etc.}$

(ii) $1+3x\sin\theta+11x^2\sin 2\theta+43x^3\sin 3\theta+\ldots$.

(17) Prove that the coefficient of x^n in the expansion of

$$e^{ax}\sin bx+e^{bx}\sin ax$$

is

$$\frac{2(a^2+b^2)^{\frac{n}{2}}}{\lfloor n}\sin\frac{n\pi}{4}\cos n\left(\frac{\pi}{4}-\tan^{-1}\frac{b}{a}\right).$$

CHAPTER VI.

RESOLUTION OF $\sin\theta$ AND $\cos\theta$ INTO FACTORS.

79. *To prove* $\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)\times \ldots$

By Arts. 38, 58, or 68 we have when n is a positive integer

$$x^{2n} - 2x^n \cos 2n\alpha + 1$$

$$= (x^2 - 2x\cos 2\alpha + 1)\left\{x^2 - 2x\cos\left(2\alpha + \frac{2\pi}{n}\right) + 1\right\}\times \ldots n \text{ factors,}$$

the last factor being

$$\left\{x^2 - 2x\cos\left(2\alpha + \frac{2(n-1)\pi}{n}\right) + 1\right\}.$$

In this result let $x = 1$, and let $2n\phi = \pi$, then

$$2(1 - \cos 2n\alpha) = 2^n(1 - \cos 2\alpha)\{1 - \cos(2\alpha + 4\phi)\}\times \ldots$$
$$\times\{1 - \cos(2\alpha + \overline{n-1}\,4\phi)\}.$$

Now $$1 - \cos 2n\alpha = 2\sin^2 n\alpha;$$

hence making this substitution and then taking the square root we obtain

$$\pm 2\sin n\alpha = 2^n \sin\alpha \sin(\alpha + 2\phi)\sin(\alpha + 4\phi)\times \ldots$$
$$\times \sin(\alpha + 2n\phi - 2\phi).$$

But $\sin(\alpha + 2n\phi - 2\phi) = \sin(\alpha + \pi - 2\phi) = \sin(2\phi - \alpha);$

$\sin\{\alpha + 2(n-2)\phi\} = \sin(4\phi - \alpha)$; and so on

Hence, when n is **odd**, we have

$$\pm 2\sin n\alpha = 2^n \sin\alpha \sin(2\phi + \alpha)\sin(2\phi - \alpha)\,.\,\sin(4\phi + \alpha)\times$$
$$\sin(4\phi - \alpha)\times \ldots \times \sin\{(n-1)\phi + \alpha\}\sin\{(n-1)\phi - \alpha\}.$$

But $\quad \sin(2\phi+\alpha)\sin(2\phi-\alpha)=\sin^2 2\phi-\sin^2\alpha.$

Hence substituting

$$\pm 2\sin n\alpha = 2^n \sin\alpha\,(\sin^2 2\phi-\sin^2\alpha)(\sin^2 4\phi-\sin^2\alpha)\times\ldots$$
$$\times\{\sin^2(n-1)\phi-\sin^2\alpha\}.$$

Next, divide both sides by $\sin\alpha$, and let α be diminished without limit; then we obtain

$$2n = 2^n\sin^2 2\phi\,.\,\sin^2 4\phi\,\sin^2 6\phi\times\ldots\times\sin^2(n-1)\,\phi.$$

Divide the first of these last two results by the second,

thus $\quad \pm\sin n\alpha = n\sin\alpha\left(1-\dfrac{\sin^2\alpha}{\sin^2 2\phi}\right)\left(1-\dfrac{\sin^2\alpha}{\sin^2 4\phi}\right)\times\ldots$

Write θ for $n\alpha$, and let n be increased while α is diminished without limit, θ remaining unchanged; then since

$$\frac{\sin^2\alpha}{\sin^2 2\phi}=\frac{\sin^2\dfrac{\theta}{n}}{\sin^2\dfrac{\pi}{n}}=\frac{\theta^2}{\pi^2}\times\frac{\sin^2\dfrac{\theta}{n}}{\left(\dfrac{\theta}{n}\right)^2}\times\frac{\left(\dfrac{\pi}{n}\right)^2}{\sin^2\dfrac{\pi}{n}},$$

therefore the limit of $\dfrac{\sin^2\alpha}{\sin^2 2\phi}$ is $\dfrac{\theta^2}{\pi^2}$; [E. 290]

and the limit of $n\sin\alpha=$ that of $n\sin\dfrac{\theta}{n}$, *i.e.* $=\theta$;

proceeding to the limit we obtain

$$\pm\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\times\ldots$$

Now, when θ lies between 0 and π, $\sin\theta$ is positive and every factor on the right-hand side is positive; when θ lies between π and 2π, $\sin\theta$ is negative and one factor only on the right-hand side is negative; and so on. Therefore the upper sign must be taken in the above result instead of the ambiguity $\pm$: and the proposition is established.

80. To resolve $\cos\theta$ into factors we proceed as in Art. 79 and obtain the identity

$$2\sin n\alpha = 2^n \sin\alpha . \sin(\alpha+2\phi)\sin(\alpha+4\phi)\times \dots \times \sin\{\alpha+(n-1)\,2\phi\}$$

In this write $\alpha+\phi$ for α, then $n\alpha$ becomes $n\alpha+n\phi$, *i e.* $n\alpha+\frac{1}{2}\pi$, and we have

$$2\cos n\alpha = 2^n\sin(\alpha+\phi)\sin(\alpha+3\phi)\sin(\alpha+5\phi)\times\dots \times\sin\{\alpha+(2n-1)\,\phi\}.$$

But $\sin\{\alpha+(2n-1)\,\phi\}=\sin(\alpha+\pi-\phi)=\sin(\phi-\alpha)$;

$$\sin\{\alpha+(2n-3)\,\phi\}=\sin(3\phi-\alpha),$$

and so on.

Hence when n is **even** we have

$$2\cos n\alpha = 2^n\sin(\phi+\alpha)\sin(\phi-\alpha)\sin(3\phi+\alpha)\sin(3\phi-\alpha)\times\dots \times\sin\{(n-1)\,\phi+\alpha\}\sin\{(n-1)\,\phi-\alpha\}$$

$$=2^n(\sin^2\phi-\sin^2\alpha)(\sin^2 3\phi-\sin^2\alpha)\times\dots \times\{\sin^2(n-1)\,\phi-\sin^2\alpha\}.$$

Whence, writing θ for $n\alpha$ as before, we obtain

$$\cos\theta=\left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\times\dots$$

[The ambiguity in the sign may be removed by the method either of Art. 79 or of Art. 81.]

*81. We can prove that the *upper* sign must be taken in each of the identities on p. 89 as follows:—

In the figure, let $ROP_0=\alpha$; produce P_0O to Q, and divide the semicircumference PQ into n equal parts P_0P_1, P_1P_2, etc. Then since $n.2\phi=\pi$, each of the angles P_0OP_1, $P_1OP_2\dots$ is equal to 2ϕ and $ROP_1=\alpha+2\phi$, $ROP_2=\alpha+4\phi$, etc. Now consider the first ambiguity on page 88.

Fig I

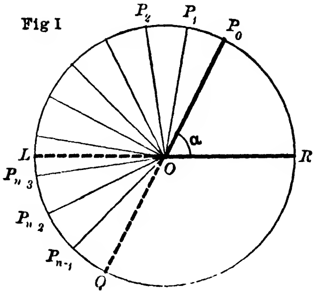
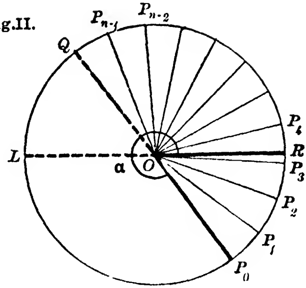


Fig.II.



I. **Let α be less than π (Fig. I.).** Now, since $\sin ROP_{n-1}$ is *negative* when P_{n-1} is *below* ROL, the product of sines on the right-hand side will be *positive* or *negative* according as the number of the points P_{n-1} P_{n-2} ... which are below ROL, is *even* or *odd*. Let r be that number. Then α, which $= ROP = LOQ$, is equal to $(r\,.\,2\phi + \lambda)$, where λ is less than 2ϕ [in Fig. I. $r = 3$ and $\lambda = LOP_{n-3}$]; hence $n\alpha = n\,(r2\phi + \lambda)$ $= r\pi + n\lambda$, where $n\lambda$ is less than π. Therefore $\sin n\alpha$ is *positive* or *negative* according as r is *even* or *odd*, that is, according as $\sin\alpha\,.\,\sin(\alpha + 2\phi)\sin(\alpha + 4\phi)\,\ldots$ is positive or negative.

II. **Let α lie between π and 2π (Fig. II.)** Then P_0 is below ROL_1, and if there are *also* r of the points P_1, P_2 . . below ROL_1, $(r+1)$ of the factors $\sin\alpha$, $\sin(\alpha+2\phi)$. .are *negative*. And in this case $\alpha = 2\pi - P_0OR = 2\pi - (r\,.\,2\phi + \lambda)$ [in Fig. II. $r = 3$ and $P_3OR = \lambda$], $\therefore\ n\alpha = 2n\pi - r\pi - n\lambda$, where $n\lambda$ is less than π. Hence $\sin n\alpha = -\sin(r\pi + \lambda)$, and therefore is *negative* or *positive* according as r is *even* or *odd*.

III. **Let α be greater than 2π**; and let $\alpha = 2m\pi + \alpha'$ where α' is less than 2π. And the proposition, being true for α' by I. and II., must also be true for α.

82. In this and the following article we give an alternative proof of the propositions of Arts. 79 and 80.

To prove $\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\times\ldots$

The proof depends on successive applications of the formula

$$2\sin A = 2\sin\frac{A}{2}\times 2\sin\left(\frac{\pi}{2}+\frac{A}{2}\right).$$

We have $2\sin\theta = 2^2\sin\frac{\theta}{2}\sin\frac{\pi+\theta}{2}$

$$\left[=2\sin\frac{\theta}{2^2}\times 2\sin\left(\frac{\pi}{2}+\frac{\theta}{2^2}\right)\times 2\sin\frac{\pi+\theta}{2^2}\times 2\sin\left(\frac{\pi}{2}+\frac{\pi+\theta}{2}\right)\right]$$

$$=2^4\sin\frac{\theta}{2^2}\sin\frac{2\pi+\theta}{2^2}\sin\frac{\pi+\theta}{2^2}\sin\frac{3\pi+\theta}{2^2}$$

$$=2^8\sin\frac{\theta}{2^3}\sin\frac{4\pi+\theta}{2^3}\sin\frac{2\pi+\theta}{2^3}\sin\frac{6\pi+\theta}{2^3}\sin\frac{\pi+\theta}{2^3}\times$$

$$\sin\frac{5\pi+\theta}{2^3}\sin\frac{3\pi+\theta}{2^3}\sin\frac{7\pi+\theta}{2^3}$$

$$=$$

$$=2^{2^n}\sin\frac{\theta}{2^n}\sin\frac{\pi+\theta}{2^n}\sin\frac{2\pi+\theta}{2^n}\sin\frac{3\pi+\theta}{2^n}\times$$

$$\times\sin\frac{(2^n-1)\pi+\theta}{2^n}\times \qquad [A.]$$

Hence since

$$\sin\frac{(2^n-1)\pi+\theta}{2^n}=\sin\frac{\pi-\theta}{2^n},\ \sin\frac{(2^n-2)\pi+\theta}{2^n}=\sin\frac{2\pi-\theta}{2^n},\ \&c.,$$

we get on rearranging the factors

$$2\sin\theta = 2^{2^n}\sin\frac{\theta}{2^n}\sin\frac{\pi+\theta}{2^n}\sin\frac{\pi-\theta}{2^n}\sin\frac{2\pi+\theta}{2^n}\sin\frac{2\pi-\theta}{2^n}\times$$

$$\sin\frac{3\pi+\theta}{2^n}\sin\frac{3\pi-\theta}{2^n}\times\ldots,$$

$$\therefore\ 2\sin\theta = 2^{2^n}\sin\frac{\theta}{2^n}\left\{\sin^2\frac{\pi}{2^n}-\sin^2\frac{\theta}{2^n}\right\}\left\{\sin^2\frac{2\pi}{2^n}-\sin^2\frac{\theta}{2^n}\right\}\times\ \ldots \text{(i)},$$

the last factor being

$$\sin\frac{2^{n-1}\pi+\theta}{2^n}=\sin\left(\frac{\pi}{2}+\frac{\theta}{2^n}\right)=\cos\frac{\theta}{2^n}.$$

In (1) divide by $\sin\frac{\theta}{2^n}$, and then put $\theta=0$, and we get

$$2^n=2^{2^n-1}\sin^2\frac{\pi}{2^n}\sin^2\frac{2\pi}{2^n}\sin^2\frac{3\pi}{2^n}\times\ldots\,. \qquad \text{(ii)}.$$

Divide (1) by (2) and we get

$$\sin\frac{\theta}{2^n}=\sin\frac{\theta}{2^n}\left\{1-\frac{\sin^2\frac{\theta}{2^n}}{\sin^2\frac{\pi}{2^n}}\right\}\left\{1-\frac{\sin^2\frac{\theta}{2^n}}{\sin^2\frac{2\pi}{2^n}}\right\}\left\{1-\frac{\sin^2\frac{\theta}{2^n}}{\sin^2\frac{3\pi}{2^n}}\right\}\times$$

Multiply by 2^n; then let n be increased without limit and we get

$$\sin\theta=\theta\left\{1-\frac{\theta^2}{\pi^2}\right\}\left\{1-\frac{\theta^2}{2^2\pi^2}\right\}\left\{1-\frac{\theta^2}{3^2\pi^2}\right\}\times\ldots\,. \qquad \text{[E. 290]}$$

83. *To prove* $\cos\theta=\left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\times$

Proceeding as in Art. 82, we obtain the result [A] viz.

$$2\sin\theta=2^{2^n}\sin\frac{\theta}{2^n}\sin\frac{\pi+\theta}{2^n}\sin\frac{2\pi+\theta}{2^n}\times \qquad \times\sin\frac{(2^n-1)\pi+\theta}{2^n}.$$

In this identity put $\frac{1}{2}\pi+\theta$ for θ, and we have

$$2\cos\theta=2^{2^n}\sin\frac{\frac{1}{2}\pi+\theta}{2^n}\sin\frac{\frac{3}{2}\pi+\theta}{2^n}\sin\frac{\frac{5}{2}\pi+\theta}{2^n}\times \qquad \times\sin\frac{(2^n-\frac{1}{2})\pi+\theta}{2^n}.$$

Hence, since

$$\sin\frac{(2^n-\frac{1}{2})\pi+\theta}{2^n}=\sin\frac{\frac{1}{2}\pi-\theta}{2^n},\quad \sin\frac{(2^n-\frac{3}{2})\pi+\theta}{2^n}=\sin\frac{\frac{3}{2}\pi-\theta}{2^n}.$$

We get on rearranging the factors

$$2\cos\theta=2^{2^n}\sin\frac{\frac{1}{2}\pi+\theta}{2^n}\sin\frac{\frac{1}{2}\pi-\theta}{2^n}\sin\frac{\frac{3}{2}\pi+\theta}{2^n}\sin\frac{\frac{3}{2}\pi-\theta}{2^n}\times \quad \ldots$$

$$\therefore\; 2\cos\theta=2^{2^n}\left\{\sin^2\frac{\frac{1}{2}\pi}{2^n}-\sin^2\frac{\theta}{2^n}\right\}\left\{\sin^2\frac{\frac{3}{2}\pi}{2^n}-\sin^2\frac{\theta}{2^n}\right\}\times\ldots \qquad \text{(i)}.$$

In this result put $\theta=0$,

$$2=2^{2^n}\sin^2\frac{\frac{1}{2}\pi}{2^n}\sin^2\frac{\frac{3}{2}\pi}{2^n}\sin^2\frac{\frac{5}{2}\pi}{2^n}\times\ldots \qquad \text{(ii)}.$$

Divide (i) by (ii); then let n be increased without limit and we get

$$\cos\theta=\left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\times\ldots\ldots \qquad \text{[E. 290]}$$

84. Many particular identities may be obtained from the results proved in Arts. 79, 80. For example, in the identity

$$2\cos n\alpha = 2^n \sin(\alpha+\phi)\sin(\alpha+3\phi)\times\ldots\times\sin(\alpha+2n\phi-\phi),$$

put $\alpha = 0$, and we have

$$1 = 2^{n-1}\sin\phi \,.\, \sin 3\phi \times \ldots \times \sin(2n-1)\phi, \text{ where } 2n\phi = \pi.$$

Again, in the identity

$$2\sin n\alpha = 2^n \sin\alpha \sin(\alpha+2\phi)\sin(\alpha+4\phi)\times\ldots \times\sin(\alpha+2n\phi-2\phi),$$

let α be diminished without limit, and we have

$$2n = 2^n \sin 2\phi \sin 4\phi \times \ldots \sin(2n\phi - 2\phi), \text{ where } 2n\phi = \pi.$$

85. The two results

$$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\times\ldots$$

and $$\cos\theta = \left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\times\ldots$$

may be proved by the aid of the following lemmas

Lemma I. *The series* $\theta - \dfrac{\theta^3}{\underline{|3}} + \dfrac{\theta^5}{\underline{|5}} -$ etc. $\equiv F(\theta)$ *is capable of being expressed as the product of an unlimited number of factors each of the form* $a_1 - \theta$.

The first n terms of the series is a rational integral function of θ of the $(2n+1)^{\text{th}}$ degree. Therefore, by the Theory of Equations, the first n terms of the series can be expressed in the form of the product of $\overline{2n+1}$ factors each of the form $a_1 - \theta$.

Hence $F(\theta)$ may be expressed in the form of (the product of $\overline{2n+1}$ factors $+ R_n$) when R_n is the remainder after n terms. This statement is true for every value of n however great.

Now $F(\theta)$ is convergent for all values of θ; therefore by choosing n sufficiently large we can make R_n as small as we please.

Therefore, by increasing n without limit, the remainder R_n vanishes; and therefore $F(\theta)$ is capable of being expressed as the product of an unlimited number of factors *each* of which is of the form $a_1 - \theta$.

[The student must notice that for different remainders the sets of factors are entirely different.]

LEMMA II. *If* a_1 *is a value of* θ *for which the series* $F(\theta)$ *vanishes then* $a_1 - \theta$ *is one of the factors of* $F(\theta)$.

For
$$F(\theta) = F(\theta) - F(a_1)$$
$$= \theta - a_1 - \frac{\theta^3}{\lfloor 3} + \frac{a_1^3}{\lfloor 3} + \text{etc. to } n \text{ terms} + R_n - R_n'$$

where R_n is the remainder of $F(\theta)$ after n terms, and R_n' is the remainder of $F(a_1)$ after n terms, where R_n and R_n' can each be made as small as we please by choosing n sufficiently large.

Hence
$$F(\theta) = (\theta - a_1)\, Q + R_n - R_n'.$$

This is true however great n may be; and since $F(\theta)$ is convergent for *all* values of θ, Q must be convergent also, when n is increased without limit; and in the limit when n is chosen sufficiently large R_n and R_n' each vanish.

Hence $$F(\theta) = (\theta - a_1)\, Q$$ Q. E. D.

NOTE. The above lemmas are true of any function of θ which is a series of ascending integral powers of θ and which is convergent for all values of θ. Therefore they are true for the series

$$1 - \frac{\theta^2}{\lfloor 2} + \frac{\theta^4}{\lfloor 4} - \dots$$

Now it has been proved in Art. 41 that

$$F(\theta) \equiv \sin\theta \text{ for all values of } \theta.$$

Also $\sin\theta$ vanishes when θ has any one of the values,

$$0,\ -\pi,\ +\pi,\ -2\pi,\ +2\pi,\ .\ -n\pi,\ +n\pi,\ \ldots;$$

Therefore $F(\theta)$ vanishes for each of these values;

Therefore $F(\theta)$ is divisible by θ, by $\pi+\theta$, by $\pi-\theta$, by $2\pi+\theta$, by $2\pi-\theta$, ... by $n\pi+\theta$, by $n\pi-\theta$, by ... and therefore by

$$\theta,\quad 1+\frac{\theta}{\pi},\quad 1-\frac{\theta}{\pi},\quad 1+\frac{\theta}{2\pi},\quad 1-\frac{\theta}{2\pi},$$

$$\therefore F(\theta)=A\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\times\ .\qquad \text{[Lemma I.]}$$

Also, $\sin\theta$ does not vanish for any other value of θ real or imaginary [Ex. Art. 30]

Therefore $F(\theta)$ has no factor of the form $a-\theta$ which is not included in the above product.

But by Lemma I. $F(\theta)$ may be expressed as the product of an unlimited number of factors every one of which factors is of the form $a-\theta$.

Therefore the factor A is a number independent of θ.

To find A we remark that when the factors

$$A\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\times\ldots$$

are multiplied out, the coefficient of θ is A, and that the coefficient of θ in $F(\theta)$ is 1; $\therefore A=1$.

Hence we have that

$$\sin\theta=F(\theta)\equiv\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\times\ldots.$$

AGAIN Lemma I. and II. are each true of the Function

$$1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \text{etc.} = \phi(\theta),$$

which function is equal to $\cos\theta$ for all values of θ.

And $\cos\theta$ vanishes when θ has any one of the values,
$-\frac{1}{2}\pi,\ +\frac{1}{2}\pi,\ -\frac{3}{2}\pi,\ +\frac{3}{2}\pi \ldots -\frac{1}{2}(2n+1)\pi,\ \ +\frac{1}{2}(2n+1)\pi,\ ..$

Therefore $\phi(\theta)$ vanishes when θ has any one of the above values.

Therefore $\phi(\theta)$ by Lemma I. is divisible

by $\frac{1}{2}\pi + \theta$, by $\frac{1}{2}\pi - \theta$, by $\frac{3}{2}\pi + \theta$, by $\frac{3}{2}\pi - \theta$, .
by $\frac{1}{2}(2n+1)\pi + \theta$, by $\frac{1}{2}(2n+1)\pi - \theta, \ldots$

and therefore by

$$1 + \frac{2\theta}{\pi},\quad 1 - \frac{2\theta}{\pi},\quad 1 + \frac{2\theta}{3\pi},\quad 1 - \frac{2\theta}{3\pi},$$

Also $\cos\theta$, and $\therefore\ \phi(\theta)$, does not vanish for any other value of θ real or imaginary.

Hence we have found all the factors of $\phi(\theta)$ and we have

$$\phi(\theta) = A\left(1 - \frac{2^2\theta^2}{\pi^2}\right)\left(1 - \frac{2^2\theta^2}{3^2\pi^2}\right)\left(1 - \frac{2^2\theta^2}{5^2\pi^2}\right) \times$$

where A is a numerical factor,

and since $\phi(\theta)$ is $1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} -$.

by multiplying out the factors and equating coefficients, we have $A = 1$.

Hence

$$\cos\theta = \phi(\theta) = \left(1 - \frac{2^2\theta^2}{\pi^2}\right)\left(1 - \frac{2^2\theta^2}{3^2\pi^2}\right) \times$$

*86. *To find the Factors of* $\cosh 2a - \cos 2\theta$, *and of* $\sinh a$, $\cosh a$.

By Art. 69, $x^{2n} - 2x^n y^n \cos 2\theta + y^{2n}$ has n factors of the form

$$x^2 - 2xy\cos\frac{2r\pi + 2\theta}{n} + y^2.$$

Let n be **odd**. The last factor is

$$x^2 - 2xy\cos\frac{2n\pi - 2\pi + 2\theta}{n} + y^2,$$

and this is equal to $x^2 - 2xy\cos\dfrac{2\pi - 2\theta}{n} + y^2$. The last factor but one is

$$x^2 - 2xy\cos\frac{2n\pi - 4\pi + 2\theta}{n} + y^2,$$

and this is equal to $x^2 - 2xy\cos\dfrac{4\pi - 2\theta}{n} + y^2$. And so on.

Hence $\qquad x^{2n} - 2x^n y^n \cos 2\theta + y^{2n}$

$$= \left(x^2 - 2xy\cos\frac{2\theta}{n} + y^2\right) \times \left(x^2 - 2xy\cos\frac{2\pi \pm 2\theta}{n} + y^2\right) \times$$

$$\ldots \times \left(x^2 - 2xy\cos\frac{2r\pi \pm 2\theta}{n} + y^2\right) \times \ldots n \text{ factors.} \qquad \text{(A.)}$$

where $\left(x^2 - 2xy\cos\dfrac{2\pi \pm 2\theta}{n} + y^2\right)$ stands for the product of the two factors

$$\left(x^2 - 2xy\cos\frac{2\pi + 2\theta}{n} + y^2\right)\left(x^2 - 2xy\cos\frac{2\pi - 2\theta}{n} + y^2\right).$$

In A. write $1 + \dfrac{a}{n}$ for x, and $1 - \dfrac{a}{n}$ for y.

Then the general form of factors on the right hand is

$$\left(1 + \frac{a}{n}\right)^2 - 2\left(1 - \frac{a^2}{n^2}\right)\cos\frac{2r\pi \pm 2\theta}{n} + \left(1 - \frac{a}{n}\right)^2,$$

that is, $$2\left(1+\frac{a^2}{n^2}\right)-2\left(1-\frac{a^2}{n^2}\right)\cos\frac{2r\pi\pm 2\theta}{n},$$

that is, $$4\sin^2\frac{r\pi\pm\theta}{n}\left\{1+\frac{a^2}{n^2}\cot^2\frac{r\pi\pm\theta}{n}\right\}.$$

In the resulting identity put $a=0$, and on the left-hand side $2-2\cos 2\theta$, that is $4\sin^2\theta$, so that A reduces to

$$4\sin^2\theta = 4\sin^2\frac{\theta}{n}\,4\sin^2\frac{\pi\pm\theta}{n}\,4\sin^2\frac{2\pi\pm\theta}{n}\times\ldots$$

Using this result to simplify the right-hand side, the identity A becomes

$$\left(1+\frac{a}{n}\right)^{2n}-2\left(1-\frac{a^2}{n^2}\right)^n\cos 2\theta+\left(1-\frac{a}{n}\right)^{2n}$$

$$=4\sin^2\theta\left(1+\frac{a^2}{n^2}\cot^2\frac{\theta}{n}\right)\times\ldots\times\left\{1+\frac{a^2}{n^2}\cot^2\frac{r\pi\pm\theta}{n}\right\}\times$$

In this let n be increased without limit, then

$$\left(1+\frac{a}{n}\right)^{2n}\ i.e.\ \left\{\left(1+\frac{a}{n}\right)^{\frac{n}{a}}\right\}^{2a}\text{ becomes }e^{2a}.\qquad\text{[Art. 2.]}$$

$$\left(1-\frac{a}{n}\right)^{2n}\ i.e.\ \left\{\left(1-\frac{a}{n}\right)^{-\frac{n}{a}}\right\}^{-2a}\text{ becomes }e^{-2a}.$$

$$\left(1-\frac{a^2}{n^2}\right)^n\ i.e.\ \left(1-\frac{a}{n}\right)^n\left(1+\frac{a}{n}\right)^n\text{ becomes }e^a\times e^{-a}\ i.e.\ 1.$$

$$\frac{a^2}{n^2}\cot^2\frac{r\pi\pm\theta}{n}\text{ becomes }\frac{a^2}{(r\pi\pm\theta)^2}.\qquad\text{Hence}$$

$$e^{2a}+e^{-2a}-2\cos 2\theta = 4\sin^2\theta\left\{1+\frac{a^2}{\theta^2}\right\}\left\{1+\frac{a^2}{(\pi\pm\theta)^2}\right\}\times$$

$$\left\{1+\frac{a^2}{(2\pi\pm\theta)^2}\right\}\times\ldots\times\left\{1+\frac{a^2}{(r\pi\pm\theta)^2}\right\}\times\ldots$$

87. In this result put $\theta = 0$; then the limit of $\sin^2\theta\left\{1+\frac{a^2}{\theta^2}\right\}$ is a^2; and we obtain

$$\left(\frac{e^a-e^{-a}}{2}\right)^2 = a^2\left(1+\frac{a^2}{\pi^2}\right)^2\left(1+\frac{a^2}{2^2\pi^2}\right)^2 \times \ldots \times \left(1+\frac{a^2}{r^2\pi^2}\right)^2 \times \ldots$$

Next put $2\theta = \pi$, and we have

$$\left(\frac{e^a+e^{-a}}{2}\right)^2 = \left(1+\frac{2^2a^2}{\pi^2}\right)^2\left(1+\frac{2^2a^2}{3^2\pi^2}\right)^2 \times \ldots \times \left(1+\frac{2^2a^2}{(2r+1)^2\pi^2}\right)^2 \times \ldots.$$

In taking the square root, since $e^a - e^{-a}$ has always the same sign as a [Art. 2], and e^a+e^{-a} is always positive, we must take the same sign for each side in each result. Hence

$$\sinh a = a\left(1+\frac{a^2}{\pi^2}\right)\left(1+\frac{a^2}{2^2\pi^2}\right)\left(1+\frac{a^2}{3^2\pi^2}\right)\times \ldots$$

$$\cosh a = \left(1+\frac{2^2a^2}{\pi^2}\right)\left(1+\frac{2^2a^2}{3^2\pi^2}\right)\left(1+\frac{2^2a^2}{5^2\pi^2}\right)\times \ldots$$

88. Since $\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \text{etc.}$

and $\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\times \ldots$

we obtain the Algebraical identity

$$1 - \frac{\theta^2}{\underline{|3}} + \frac{\theta^4}{\underline{|5}} - \text{etc.} = \left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\times \ldots \quad \text{(I.)}$$

Similarly, from the two expressions equivalent to $\cos\theta$, we obtain

$$1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \text{etc.} = \left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\times \ldots \quad \text{(II.)}$$

I. and II. are identities, and are true for all values of θ. Therefore they are true if we write ϕ for θ^2; also they are true when in the resulting identities for ϕ, we write $-\phi$

In the resulting identities we may write a^2 for ϕ, then

$$1+\frac{a^2}{\lfloor 3}+\frac{a^4}{\lfloor 5}+\ldots=\left(1+\frac{a^2}{\pi^2}\right)\left(1+\frac{a^2}{2^2\pi^2}\right)\left(1+\frac{a^2}{3^2\pi^2}\right)\times$$

$$1+\frac{a^2}{\lfloor 2}+\frac{a^4}{\lfloor 4}+\quad\left(1+\frac{2^2a^2}{\pi^2}\right)\left(1+\frac{2^2a^2}{3^2\pi^2}\right)\left(1+\frac{2^2a^2}{5^2\pi^2}\right)\times$$

By the above artifice the results of Art. 87 and of Arts. 79, 80 may be deduced the one from the other without the introduction of $\sqrt{(-1)}$.

89 Many results may be obtained from the identities

$$1-\frac{\theta^2}{\lfloor 3}+\frac{\theta^4}{\lfloor 5}-\text{etc}\quad\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\times$$

$$1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\text{etc}=\left(1-\frac{2^2\theta^2}{\pi^2}\right)\left(1-\frac{2^2\theta^2}{3^2\pi^2}\right)\left(1-\frac{2^2\theta^2}{5^2\pi^2}\right)\times\ldots$$

EXAMPLE 1. *Prove that* $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\quad=\frac{\pi^2}{6}$.

From first of the above identities we have

$$\log\left(1-\frac{\theta^2}{\lfloor 3}+\frac{\theta^4}{\lfloor 5}-\text{etc.}\right)=\log\left(1-\frac{\theta^2}{\pi^2}\right)+\log\left(1-\frac{\theta^2}{2^2\pi^2}\right)$$
$$+\log\left(1-\frac{\theta^2}{3^2\pi^2}\right)+\text{etc.}$$

Expanding each of these logarithms by Art. 4, we have

$$\left(\frac{\theta^2}{\lfloor 3}-\frac{\theta^4}{\lfloor 5}+\quad.\right)+\tfrac{1}{2}\left(\frac{\theta^2}{\lfloor 3}-\frac{\theta^4}{\lfloor 5}+\quad.\right)^2+\text{etc.}$$
$$=\left(\frac{\theta^2}{\pi^2}+\tfrac{1}{2}\frac{\theta^4}{\pi^4}+\text{etc.}\right)+\left(\frac{\theta^2}{2^2\pi^2}+\tfrac{1}{2}\frac{\theta^4}{2^4\pi^4}+\text{etc.}\right)$$
$$+\left(\frac{\theta^2}{3^2\pi^2}+\tfrac{1}{2}\cdot\frac{\theta^4}{3^4\pi^4}+\text{etc.}\right)+\text{etc.}$$
$$=\frac{\theta^2}{\pi^2}\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\quad\right)+\frac{\theta^4}{2\pi^4}\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\quad.\right).$$

In this identity we may equate the coefficients of the various powers of θ^2. Hence

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\quad=\frac{\pi^2}{1\,.\,2\,.\,3}.$$

*EXAMPLE 2. *To prove* $\cot\theta = \dfrac{1}{\theta} - \dfrac{2\theta}{\pi^2-\theta^2} - \dfrac{2\theta}{2^2\pi^2-\theta^2} - \dfrac{2\theta}{3^2\pi^2-\theta^2} - \ldots$

By Art. 79, $\log\sin\theta = \log\theta + \log\left(1-\dfrac{\theta^2}{\pi^2}\right) + \log\left(1-\dfrac{\theta^2}{2^2\pi^2}\right) + \ldots$ I.

The required result may be obtained by writing $\theta+h$ for θ in this identity, expanding each term in ascending powers of θ, and then equating the coefficient of h on each side Now

$$\log\sin(\theta+h) = \log\sin\theta + h\cot\theta - \tfrac{1}{2}h^2\operatorname{cosec}^2\theta - \text{etc. as in Art. 109}$$

$$\log(\theta+h) = \log\left\{\theta\left(1+\frac{h}{\theta}\right)\right\} = \log\theta + \frac{h}{\theta} - \frac{1}{2}\frac{h^2}{\theta^2} + \ldots \qquad \text{[Art. 4.]}$$

$$\log\left\{1-\frac{(\theta+h)^2}{r^2\pi^2}\right\} = \log\left\{1-\frac{\theta^2}{r^2\pi^2}-\frac{2h\theta}{r^2\pi^2}-\frac{h^2}{r^2\pi^2}\right\}$$

$$=\log\left\{\left(1-\frac{\theta^2}{r^2\pi^2}\right)\left(1-\frac{2h\theta}{r^2\pi^2-\theta^2}-\frac{h^2}{r^2\pi^2-\theta^2}\right)\right\}$$

$$=\log\left(1-\frac{\theta^2}{r^2\pi^2}\right)+\log\left(1-\frac{2h\theta}{r^2\pi^2-\theta^2}-\frac{h^2}{r^2\pi^2-\theta^2}\right)$$

$$=\log\left(1-\frac{\theta^2}{r^2\pi^2}\right)-\frac{2h\theta}{r^2\pi^2-\theta^2}-h^2\frac{r^2\pi^2+\theta^2}{(r^2\pi^2-\theta^2)^2}-\text{etc. [Art. 4.]}$$

Hence, making these substitutions in

$$\log\sin(\theta+h) = \log(\theta+h) + \log\left\{1-\frac{(\theta+h)^2}{\pi^2}\right\} + \text{etc.},$$

and equating the coefficients of h, we obtain the required result.

EXAMPLES. XXXI.

Prove the following statements:

(1) $\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\dfrac{1}{7^2}+\ldots = \frac{1}{8}\pi^2.$

(2) $\dfrac{1}{1^4}+\dfrac{1}{2^4}+\dfrac{1}{3^4}+\dfrac{1}{4^4}+\ldots = \frac{1}{90}\pi^4.$

(3) $\dfrac{1}{1^4}+\dfrac{1}{3^4}+\dfrac{1}{5^4}+\dfrac{1}{7^4}+\ldots = \frac{1}{96}\pi^4.$

(4) The sum of the products of the squares of the reciprocals of every pair of positive integers is $\frac{1}{120}\pi^4$

(5) $\dfrac{\pi}{2} = \dfrac{2^2}{1\cdot 3}\cdot\dfrac{4^2}{3\cdot 5}\cdot\dfrac{6^2}{5\cdot 7}\ldots$

(6) When n is even

(i) $$2^{\frac{n}{2}-1}\sin\frac{\pi}{2n}\cdot\sin\frac{3\pi}{2n}\times\ldots\times\sin\frac{n-1}{2n}\pi=1$$

(ii) $$2^{\frac{n}{2}-1}\cos\frac{\pi}{n}\cdot\cos\frac{2\pi}{n}\times\ldots\times\cos\frac{n-4}{2n}\pi\cdot\cos\frac{n-2}{2n}\pi=\sqrt{n}$$

(7) $$\frac{\sin(a-\theta)}{\sin a}=\left(1-\frac{\theta}{a}\right)\left(1+\frac{\theta}{\pi-a}\right)\left(1-\frac{\theta}{\pi+a}\right)\left(1+\frac{\theta}{2\pi-a}\right)$$
$$\times\left(1-\frac{\theta}{2\pi+a}\right)\left(1+\frac{\theta}{3\pi-a}\right)\left(1-\frac{\theta}{3\pi+a}\right)\times$$

(8) $$\frac{\sin(a+\theta)}{\sin a}=\left(1+\frac{\theta}{a}\right)\left(1-\frac{\theta}{\pi-a}\right)\left(1+\frac{\theta}{\pi+a}\right)\left(1-\frac{\theta}{2\pi-a}\right)\times$$

(9) $$\frac{\cos(a-\theta)}{\cos a}=\left(1+\frac{2\theta}{\pi-2a}\right)\left(1-\frac{2\theta}{\pi+2a}\right)\left(1+\frac{2\theta}{3\pi-2a}\right)\times$$
$$\left(1-\frac{2\theta}{3\pi+2a}\right)\left(1+\frac{2\theta}{5\pi-2a}\right)\left(1-\frac{2\theta}{5\pi+2a}\right)\times$$

(10) $$\frac{\cos(a+\theta)}{\cos a}=\left(1-\frac{2\theta}{\pi-2a}\right)\left(1+\frac{2\theta}{\pi+2a}\right)\left(1-\frac{2\theta}{3\pi-2a}\right)\times$$

(11) $$\frac{\cos\theta+\cos a}{1+\cos a}=\frac{\cos\frac{1}{2}(a-\theta)\cos\frac{1}{2}(a+\theta)}{\cos\frac{1}{2}a\cos\frac{1}{2}a}$$
$$=\left\{1-\frac{\theta^2}{(\pi\pm a)^2}\right\}\left\{1-\frac{\theta^2}{(3\pi\pm a)^2}\right\}\left\{1-\frac{\theta^2}{(5\pi\pm a)^2}\right\}\times$$

where $1-\frac{\theta^2}{(\pi\pm a)^2}$ stands for $\left\{1-\frac{\theta^2}{(\pi+a)^2}\right\}\left\{1-\frac{\theta^2}{(\pi-\theta)^2}\right\}$.

(12) $$\frac{\cos\theta-\cos a}{1-\cos a}=\left(1-\frac{\theta^2}{a^2}\right)\left\{1-\frac{\theta^2}{(2\pi\pm a)^2}\right\}\left\{1-\frac{\theta^2}{(4\pi\pm a)^2}\right\}$$

(13) $$\frac{\sin\theta+\sin a}{\sin a}$$
$$=\left(1+\frac{\theta}{a}\right)\left(1+\frac{\theta}{\pi-a}\right)\left(1-\frac{\theta}{\pi+a}\right)\left(1+\frac{\theta}{2\pi+a}\right)\left(1-\frac{\theta}{2\pi-a}\right)\times$$

(14) $$\frac{\sin a-\sin\theta}{\sin a}=\left(1-\frac{\theta}{a}\right)\left(1-\frac{\theta}{\pi-a}\right)\left(1+\frac{\theta}{\pi+a}\right)\left(1-\frac{\theta}{2\pi+a}\right)\times$$

(15) From the result of Ex. (11) deduce the factors of

$$\cosh x+\cos a.$$

(16) From the result of Ex. (12) deduce the factors of

$$\cos x - \cos a.$$

(17) From the result of Ex. (7) prove that

$$\cot a = \frac{1}{a} - \frac{2a}{\pi^2 - a^2} - \frac{2a}{2^2\pi^2 - a^2} - \frac{2a}{3^2\pi^2 - a^2} -$$

(18) $\tan x = \frac{2}{\pi - 2x} - \frac{2}{\pi + 2x} + \frac{2}{3\pi - 2x} - \frac{2}{3\pi + 2x} + \text{etc.}$

(19) $\frac{1}{\sin a} = \frac{1}{a} + \frac{2a}{\pi^2 - a^2} - \frac{2a}{2^2\pi^2 - a^2} + \frac{2a}{3^2\pi^2 - a^2} -$

(20) $\frac{1}{4\cos a} = \frac{\pi}{\pi^2 - 2^2a^2} - \frac{3\pi}{3^2\pi^2 - 2^2a^2} + \frac{5\pi}{5^2\pi^2 - 2^2a^2} -$.

(21) Since $e^{2x} + e^{-2x} - 2\cos 2a = 2\cos 2ix - 2\cos 2a$

$$= 4\sin(a + ix)\sin(a - ix),$$

deduce from the factors of $\sin\theta$ those of $\cosh 2x - \cos 2a$.

(22) $\frac{1}{x^2 + 1^2} + \frac{1}{x^2 + 3^2} + \frac{1}{x^2 + 5^2} + \text{etc.} = \frac{\pi}{4x}\,\frac{e^{\frac{\pi x}{2}} - e^{-\frac{\pi x}{2}}}{e^{\frac{\pi x}{2}} + e^{-\frac{\pi x}{2}}}.$

(23) $\frac{1}{1^2 + x^2} + \frac{1}{2^2 + x^2} + \frac{1}{3^2 + x^2} + \text{etc.} = \frac{\pi}{2x}\cdot\frac{e^{\pi x} + e^{-\pi x}}{e^{\pi x} - e^{-\pi x}} - \frac{1}{2x^2}.$

(24) $$\left(1 + \frac{2}{1 + 1^2} + \frac{2}{1 + 2^2} + \frac{2}{1 + 3^2} + \quad\right)$$
$$\times\left(\frac{1}{4 + 1^2} + \frac{1}{4 + 3^2} + \frac{1}{4 + 5^2} + \quad\right) = \frac{\pi^2}{8}.$$

(25) $$\operatorname{cosec}^2\theta = \frac{1}{\theta^2} + 2\frac{\pi^2 + \theta^2}{(\pi^2 - \theta^2)^2} + 2\frac{2^2\pi^2 + \theta^2}{(2^2\pi^2 - \theta^2)^2}$$
$$+ 2\frac{3^2\pi^2 + \theta^2}{(3^2\pi^2 - \theta^2)^2} +$$

(26) Find the factors of

$$1 - \frac{\theta}{\underline{|3}} + \frac{\theta^3}{\underline{|5}} - \frac{\theta^4}{\underline{|7}} +$$

(27) Find the factors of

$$1 + \frac{a}{\underline{|2}} + \frac{a^2}{\underline{|4}} +$$

CHAPTER VII.

ON THE INTERPRETATION OF $\sqrt{-1}$.

90. IN Algebra, one of the interpretations which we give to the symbol $-$ is, that when $+$ a certain *number* of linear units indicates a distance in some chosen direction, then $-$ a certain number of linear units indicates a distance in the same line but in the *opposite direction.*

It is also shewn that positive and negative numbers having this interpretation obey the laws of algebra.

Hence, when working out with algebraical symbols a problem concerning distance, we interpret the symbol $-$ to indicate a complete change of direction.

91. Now quantities which contain $\sqrt{-1}$ as a factor are in some way different from quantities which do not contain this factor.

For when $A + \sqrt{-1}B = a + \sqrt{-1}b$ then $A = a$ and $B = b$.

We want then an interpretation of the effect of multiplying a distance by $\sqrt{-1}$.

We have seen that the effect of multiplying a distance by -1 may be said to be, to *turn* the distance *through two right angles.*

And whatever interpretation we give to $\sqrt{-1}$ it must be such that the multiplication of a distance by

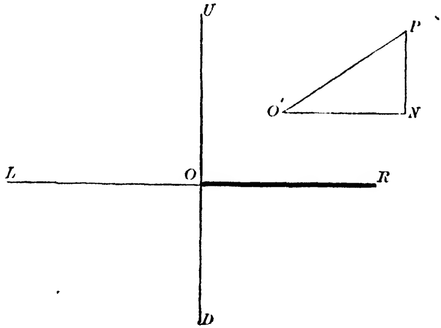
$$\sqrt{-1} \times \sqrt{-1},$$

that is, by -1, must have the effect of *turning* a distance through *two* right angles.

Hence, it seems worth while to consider how far we may interpret the effect of multiplying a distance by $\sqrt{-1}$ by supposing that it *turns* that distance *through* one right angle.

92. Accordingly we propose the following interpretation; having chosen a direction OR as the positive direction, we produce RO backwards to L, and call OL the negative direction.

We then draw OU perpendicular to OR, and call OU the $\sqrt{-1}$ direction; producing UO in the opposite direction to D we have OD the $-\sqrt{-1}$ direction.



Now take any line $O'P$ in the direction making the angle a with OR; draw PN parallel to UO and $O'N$ parallel to OR.

Now suppose the length of $O'P$ is r feet, then

$$NP = r \sin a \text{ feet,}$$

and
$$O'N = r \cos a \text{ feet.}$$

To indicate the position of P relative to O' we may say that it is the distance

$$r(\cos a + \sqrt{-1} \sin a) \text{ feet}$$

That is, if from O' we go $r \cos a$ feet $+ \sqrt{-1}\, r \sin a$ feet we shall arrive at P.

We notice that when we have done this we shall have arrived at a point P whose distance in a straight line from O' is r feet.

Thus $(\cos a + \sqrt{-1} \sin a)$ may be said to *turn* a distance *through the angle* a

For, $(\cos a + \sqrt{-1} \sin a)\, r$ feet indicates a distance r feet from the initial point in the direction making the angle a with the chosen positive direction.

NOTE. The chief fundamental rules of Algebra are

$$a + b = b + a, \quad a - b = -b + a \qquad \text{(i)},$$

$$\left.\begin{aligned} a + (+b) &= a + b, \quad a + (-b) = a - b \\ a - (+b) &= a - b; \quad a - (-b) = a + b \end{aligned}\right\} \qquad \text{(ii)},$$

$$\left.\begin{aligned} (+a) \times (+b) &= +ab; \quad (+a) \times (-b) = -ab \\ (-a) \times (+b) &= -ab; \quad (-a) \times (-b) = +ab \end{aligned}\right\} \qquad \text{(iii)},$$

$$ab = ba \qquad \text{(iv)}.$$

The geometrical application of Algebra may be explained thus:

The unit is a distance; say *one foot.*

a considered as a symbol of geometrical operation indicates that a line is to be extended to a times its original length, $+$ indicates that the line is to be drawn in a certain chosen direction; hence,

$+a$ feet indicates the following operation performed on the unit.

From a chosen starting point called the initial point we draw a line one foot long in a chosen direction called the positive direction, and this line is then produced to a times its length.

The symbol $-$ indicates that the line to which it is applied has been turned through two right angles.

The symbol $\sqrt{(-1)}$ indicates that the line to which it is applied has been turned through one right angle.

Now provided we interpret a group of operations performed upon the unit as indicating *the transferring of the initial point from its chosen position to another position*, it will be found that the symbols interpreted geometrically as above obey all the fundamental laws of Algebra.

We give the proof of two of these laws, leaving the rest as an Exercise to the student.

To shew that $\quad (-a)\times(-b)=+ab.$

$(-b)$ performed on the unit carries the initial point b feet in the positive direction and then turns it through two right angles.

$(-b)$ performed on this line a feet long in the negative direction extends it to b times its length, i.e. makes it ab feet and turns it through two right angles; it will be seen that by this process the initial point has been carried ab feet from its chosen position in the positive direction; that is

$$(-b)\times(-a)=+ab.$$

To shew that $\quad a+\sqrt{(-1)}\,b=\sqrt{(-1)}\,b+a.$

$a+\sqrt{(-1)}\,b$ performed on the unit carries the initial point a feet in the positive direction and then b feet farther in the direction at right angles to it.

$\sqrt{(-1)}\,b+a$ performed on the unit carries the initial point b feet in the direction perpendicular to the positive direction and then a feet farther in the positive direction.

In the first case the initial point is carried along one pair of sides of a rectangle, and in the second along the opposite pair of sides, landing the initial point at the same spot in each case;

$$\therefore\; a+\sqrt{(-1)}\,b=\sqrt{(-1)}\,b+a.$$

93. Consider any complex expression

$$a + ib.$$

Taking a foot as our unit distance $(a + ib)$ means go a feet in the positive direction, and b feet in the direction perpendicular to it.

Let $$a + ib = r(\cos\alpha + i\sin\alpha),$$

then $$r = \sqrt{a^2 + b^2},$$

$$\cos\alpha = \frac{a}{r} \text{ and } \sin\alpha = \frac{b}{r}.$$

In the figure on page 104, if $O'N = a$ feet, $NP = b$ feet, then $O'P = r$ feet.

Thus $\sqrt{a^2 + b^2}$ the modulus of the expression $a + ib$, may be said to indicate the *resultant distance* r.

94. We have said that the factor

$$\cos\alpha + i\sin\alpha$$

turns a distance through the angle α.

Therefore the factor

$$(\cos\alpha + i\sin\alpha) \times (\cos\beta + i\sin\beta)$$

turns a distance through the angle $(\alpha + \beta)$ and therefore should be equivalent to the factor

$$\cos(\alpha + \beta) + i\sin(\alpha + \beta).$$

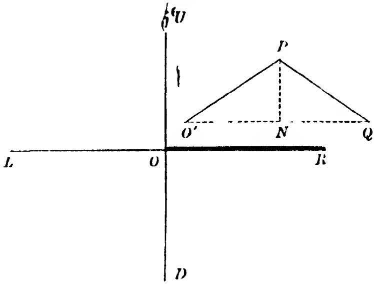
This is true by De Moivre's Theorem.

Thus our interpretation of i gives us a complete interpretation of De Moivre's Theorem.

95. Let us suppose that the factor $e^{i\theta}$ turns a distance through the angle θ.

Then $(e^{i\theta} + e^{-i\theta})\,a$ feet would mean

Take any point O'; draw $O'P = a$ feet in the di-



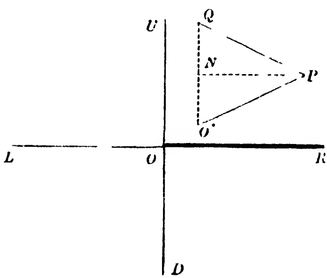
rection θ; from P draw $PQ = O'P$ in the direction $-\theta$

Now by the geometry of the figure, $O'Q$ is equal to $a \times 2\cos\theta$ feet in the *positive* direction.

So that the factor $e^{\iota\theta} + e^{-\iota\theta}$ is equivalent to the factor

$$2 \cos \theta.$$

Again $(e^{\iota\theta} - e^{-\iota\theta}) \times a$ feet would mean Go a distance $O'P = a$ feet in the direction θ and then from P go a distance $-PQ$ in the direction $-\theta$ and by



the geometry of the figure $O'Q$ is equal to $a \times 2 \sin\theta$ feet *in the* $\sqrt{-1}$ *direction.*

So that the factor $e^{\iota\theta} - e^{-\iota\theta}$ is equivalent to the factor

$$2 \sin \theta \times \sqrt{-1}.$$

We have now some notion of the meaning of the symbol = in such an expression as $e^{\iota\theta}+e^{-\iota\theta}=2\cos\theta$; $e^{\iota\theta}+e^{-\iota\theta}$ may be said to transfer its initial point to the same final position as $2\cos\theta$ does but by a different path.

EXAMPLES. XXXII.

(1) Prove that the two expressions

$$r(\cos\alpha+\iota\sin\alpha)+r'(\cos\beta+\iota\sin\beta)$$

and

$$\rho(\cos\theta+\iota\sin\theta)$$

where $\rho^2=r^2+r'^2-2rr'\cos(\alpha-\beta)$ and $\tan\theta=\dfrac{r\cos\alpha+r'\cos\beta}{r\sin\alpha+r'\sin\beta}$ are equivalent algebraically and geometrically.

(2) Prove that the factor $e^{2\iota r\pi}$ where r is a whole number, is a factor which does not alter the quantity multiplied.

(3) From Arts. 49 and 83 deduce the following general proposition. Any formula which is true for all values of θ, which contains terms involving *odd* integral powers of the sine of any multiple of θ, cannot contain any term which involves an *even* power of the sine of any multiple of θ.

CHAPTER VIII.

THE RULE OF PROPORTIONAL DIFFERENCES,

OTHERWISE CALLED

THE THEORY OF PROPORTIONAL PARTS.

96. The logarithms in this Chapter are Common Logarithms.

In the Elementary Trigonometry a Rule called the Rule of Proportional Differences was given, and it was shown that, assuming the Rule to be *practically* true, we are enabled to use Tables of a more moderate size than would otherwise be necessary. [Cf. E. 218—222.]

The RULE is as follows The differences between three numbers are proportional to the corresponding differences between the logarithms of those numbers, provided the differences between the numbers are small compared with the numbers.

In this Chapter we shall prove that this Rule of Proportional Differences is practically true as applied to the Table of Logarithms of Numbers, and that it is also practically true *in general* as applied to Tables of Trigonometrical Ratios and their Logarithms.

97. The Rule as applied to the Tables of Logarithms may be stated thus:

Let n be any number greater than 10000 and less than 100000; let d be any number not greater than unity; then as far as seven places of decimals the following proportion is true:

$$\frac{\log(n+d) - \log n}{\log(n+1) - \log n} = \frac{d}{1}.$$

98. *To prove the Rule for the Table of* **common logarithms**

We have

$$\log(n+d) = \log n\left(1+\frac{d}{n}\right) = \log n + \log\left(1+\frac{d}{n}\right)$$

$$= \log n + \mu\left\{\frac{d}{n} - \frac{1}{2}\frac{d^2}{n^2} + \frac{1}{3}\frac{d^3}{n^3} - \ldots\right\}. \quad \text{[Art. 4.]}$$

Let n be not less than 10000 and d not greater than 1; also μ the modulus [Art. 7] is ·4342945.. Hence μ is less than ·5, $\frac{d}{n}$ is not greater than ·0001. Therefore $\frac{1}{2}\frac{\mu d^2}{n^2}$ is not greater than $\frac{1}{4}(\cdot 0001)^2$, *i.e.* not greater than ·0000000025, $\frac{\mu}{3}\frac{d^3}{n^3}$ is much less than this

Hence *at least as far as seven decimal places*

$$\log(n+d) - \log n = \frac{\mu d}{n}.$$

Similarly $$\log(n+1) - \log n = \frac{\mu}{n}.$$

Therefore $$\frac{\log(n+d) - \log n}{\log(n+1) - \log n} = \frac{d}{1};$$

which proves the Rule.

99. *To prove the Rule for the Table of natural* **sines.**

That is, *To prove that if a be any angle, and δ, δ' angles not greater than* 1′, *then*

$$\frac{\sin(a+\delta)-\sin a}{\sin(a+\delta')-\sin a}=\frac{\delta}{\delta'}$$

as far as seven decimal places.

We have $\sin(a+\delta)=\sin a\cos\delta+\cos a\sin\delta$

$$=\sin a\left\{1-\tfrac{1}{2}\delta^2+\ldots\right\}+\cos a\left\{\delta-\tfrac{1}{6}\delta^3+\ldots\right\}\ \text{[Art. 41.]}$$

$$=\sin a+\delta\cos a-\tfrac{1}{2}\delta^2\sin a-\tfrac{1}{6}\delta^3\cos a+\ldots.$$

δ is here the circular measure of an angle not greater than 1′, $\therefore$ δ is not greater than ·0003 [E. Ex. x. 17], $\therefore$ $\frac{1}{2}\delta^2$ is not greater than ·00000005 and $\sin a$ is not greater than 1. Hence, as far as seven places of decimals in the value of the sines,

$$\sin(a+\delta)-\sin a=\delta\cos a.$$

Similarly $\sin(a+\delta')-\sin a=\delta'\cos a$

Therefore $$\frac{\sin(a+\delta)-\sin a}{\sin(a+\delta')-\sin a}=\frac{\delta}{\delta'}=\frac{n}{n'},$$

where n and n' are the numbers of seconds in the angles δ and δ' respectively.

Thus the rule of proportional difference is true as applied to the Table of natural sines.

100. It must be observed however, that when a is nearly $\frac{1}{2}\pi$, $\cos a$ is very small, and $\sin a$ approaches unity; so that when a is nearly $\frac{1}{2}\pi$, $\delta\cos a$ is comparable with

$$\tfrac{1}{2}\delta^2\sin a.$$

Hence, when a is within a few minutes of 90° we cannot neglect $-\frac{1}{2}\delta^2\sin a$ *in comparison with* $\delta\cos a$.

Therefore in this case we must say that

$$\sin(\alpha+\delta)-\sin\alpha=\delta\cos\alpha-\tfrac{1}{2}\delta^2\sin\alpha.$$

Hence, the differences between the sines of two angles which are each nearly 90^0, are *not* approximately proportional to the difference between the angles.

The differences are then said to be **irregular**.

It must be noticed, however, that we have proved that $\frac{1}{2}\delta^2\sin\alpha$ is less than ·00000005, and as *we are neglecting figures after seven places* of decimals the term $-\frac{1}{2}\delta^2\sin\alpha$ does not affect the result.

A quantity whose measure when expressed in terms of the unit under consideration is less than ·0000001 is said to be **insensible.**

Consequently, in the case of sines, when the differences become *irregular* they are at the same time *insensible*. This insensibility gives rise to a serious practical difficulty. See Arts. 114, 127.

101. *The case of the Natural* **cosines.**

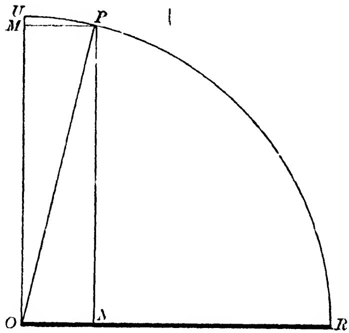
Since $\cos(\alpha+\delta)=\cos\alpha\cos\delta-\sin\alpha\sin\delta$

$$=\cos\alpha-\delta\sin\alpha-\tfrac{1}{2}\delta^2\cos\alpha+\tfrac{1}{6}\delta^3\sin\alpha+\dots,$$

we may prove as in Art. 99 that the rule of proportional differences is practically applicable to the cosines of angles when the differences between the angles are less than one minute.

Also, we may prove as in Art. 100 that when α is nearly zero the differences between the cosines are **irregular,** but **insensible.**

102. The above results may be explained geometrically thus:



Let RPU be the quadrant of a circle, radius unity. Let ROP be an angle nearly 90°, draw PN, PM perpendiculars to OR and OU.

Then the arc UP and the line PM *approach to coincidence* as the angle POU is diminished. In other words the path of P differs very little from the line PM.

Hence, when ROP is nearly a right angle the differences in PN, as the angle approaches a right angle, are small when compared with the differences in the arc RP. Hence it is said that the differences in $\sin\theta$ when θ is nearly $\frac{1}{2}\pi$ are insensible.

The irregularity in the differences is caused by their smallness and consequently it is of little importance.

Also since the measure of PN is the cosine of the angle UOP, we can see from the same figure the cause of the insensibility of difference in the cosine of a *small* angle.

103. The case of the **tangent** may be discussed as follows:

$$\tan(\theta+\delta) = \frac{\sin(\theta+\delta)}{\cos(\theta+\delta)}$$
$$= \frac{\sin\theta + \delta\cos\theta - \frac{1}{2}\delta^2\sin\theta - \text{etc}}{\cos\theta - \delta\sin\theta - \frac{1}{2}\delta^2\cos\theta + \text{etc.}}$$
$$= \frac{\sin\theta\,(1+\delta\cot\theta - \frac{1}{2}\delta^2 - \text{etc.})}{\cos\theta\,(1-\delta\tan\theta - \frac{1}{2}\delta^2 + \text{etc.})}$$

hence, neglecting higher powers of δ than the second, we have

$$\tan(\theta+\delta) = \tan\theta\,(1+\delta\cot\theta - \tfrac{1}{2}\delta^2)\{1-(\delta\tan\theta + \tfrac{1}{2}\delta^2)\}^{-1}$$
$$= \tan\theta\,(1+\delta\cot\theta - \tfrac{1}{2}\delta^2)(1+\delta\tan\theta + \tfrac{1}{2}\delta^2 + \delta^2\tan^2\theta)$$
$$= \tan\theta\,\{1+\delta(\cot\theta+\tan\theta) + \delta^2(1+\tan^2\theta)\}$$
$$= \tan\theta + \delta\sec^2\theta + \delta^2\frac{\sin\theta}{\cos^3\theta}.$$

Hence, unless $\sin\theta\sec^3\theta$ is large, we have

$$\tan(\theta+\delta) - \tan\theta = \delta\sec^2\theta,$$

which proves the rule in this case.

104. Suppose that the Table of tangents is calculated for every minute. Then the largest value of δ (as in Art. 99) is ·0003 nearly. Hence the greatest value of $\delta^2\sin\theta\sec^3\theta$ is (·00000009) $\sin\theta\sec^3\theta$ nearly. So that when θ is greater than $\frac{1}{4}\pi$ we are liable to an error in the seventh place of decimals. Hence the Rule is not true for tables of tangents calculated for every minute, when the angle is between 45° and 90°.

105. It follows immediately from Art. 103, since the **cotangent** of an angle is equal to the tangent of its complement, that the rule must not be used for a table of cotangents, calculated for every minute, when the angle lies between 0 and 45°.

106. The case of the **secant.**

$$\sec(\theta+\delta)=\frac{1}{\cos(\theta+\delta)}=\frac{1}{\cos\theta\,(1-\delta\tan\theta-\frac{1}{2}\delta^2+\ldots)}$$
$$=\sec\theta\,\{1+\delta\tan\theta+\delta^2(\tfrac{1}{2}+\tan^2\theta)+\ldots\}.$$

Hence, neglecting powers of δ above the second,

$$\sec(\theta+\delta)-\sec\theta=\delta\sin\theta\sec^2\theta+\delta^2(\tfrac{1}{2}+\tan^2\theta)\sec\theta.$$

Whence it may be shewn that the differences are *irregular* but *insensible* when the angles are very small, and they are irregular but large when the angle approaches a right angle, and that with these exceptions the Rule of Proportional Differences is true.

107. The case of the **cosecant.**

The cosecant of an angle is the secant of its complement.

108. The Rule of Proportional Differences is particularly of importance in *practical work.* In Practice the **Logarithms of the Trigonometrical Ratios** are more often used than the Ratios themselves. It is therefore particularly important to consider whether the Rule is true with respect to the Logs of the Trigonometrical Ratios.

109. *To consider the case of the* **L sine.**

$$\log\sin(\theta+\delta)=\log\{\sin\theta+\delta\cos\theta-\tfrac{1}{2}\delta^2\sin\theta-\text{etc.}\}$$
$$=\log\sin\theta\,\{1+\delta\cot\theta-\tfrac{1}{2}\delta^2-\text{etc.}\}$$
$$=\log\sin\theta+\log\{1+\delta(\cot\theta-\tfrac{1}{2}\delta-\text{etc.})\}$$
$$=\log\sin\theta+\mu\delta(\cot\theta-\tfrac{1}{2}\delta-\ldots)-\tfrac{1}{2}\mu\delta^2(\cot\theta-\tfrac{1}{2}\delta-\ldots)^2+\ldots$$
$$=\log\sin\theta+\mu\delta\cot\theta-\tfrac{1}{2}\mu\delta^2\{1+\cot^2\theta\}+\ldots$$
$$=\log\sin\theta+\mu\delta\cot\theta-\tfrac{1}{2}\mu\delta^2\operatorname{cosec}^2\theta+\ldots$$

Hence, omitting higher powers of δ than δ^2, we have

$$L\sin(\theta+\delta)-L\sin\theta=\mu\delta\cot\theta-\tfrac{1}{2}\mu\delta^2\operatorname{cosec}^2\theta.$$

If the Tables are calculated for every ten seconds, δ is not greater than ·00005, and therefore, unless $\cot\theta$ is small or $\operatorname{cosec}^2\theta$ large, we have $L\sin(\theta+\delta) - L\sin\theta = \mu\delta\cot\theta$ as far as seven places of decimals, which proves the Rule to be *generally* true

110. When θ is small, $\operatorname{cosec}\theta$ is large. Suppose that the Tables give the L sin of every $10''$. Then δ is not greater than the circular measure of $10''$, which is ·0000484. . and μ is not greater than $\frac{1}{2}$. Hence $\frac{1}{2}\mu\delta^2\operatorname{cosec}^2\theta$ is not greater than $\dfrac{6\operatorname{cosec}^2\theta}{10^{10}}$. In order that this may not affect the *seventh* decimal place $6\operatorname{cosec}^2\theta$ must not be greater than 10^3, that is θ must not be less than about 5^0.

Also when θ is small, $\cot\theta$ is large. Hence when the angles are small, the differences of consecutive L sines are irregular and they are *not* insensible; so that the rule of Proportional Parts does not apply to the L sine when the angle is less than 5^0.

111. When θ is nearly a right angle $\cot\theta$ is small and $\operatorname{cosec}\theta$ approaches unity.

Hence when the angles are nearly right angles, the differences of consecutive L sines are irregular but they are at the same time insensible.

112. *The case of the Table of* **L cosines.**

Similar conclusions concerning $\log\cos\theta$ may be inferred from the formula

$$\log\cos(\theta+\delta) = \log\cos\theta - \mu\delta\tan\theta - \tfrac{1}{2}\mu\delta^2\sec^2\theta + \ldots$$

The differences in this case will be irregular and large when θ is nearly a right angle, and irregular and insensible when θ is nearly zero. This is also clear because the sine of an angle is the cosine of its complement.

113. We find then that the Rule of proportional differences cannot be applied to interpolate between the L sines of angles which differ by $10''$, when the angle is less than 5°. Three methods have been proposed to replace the Rule.

I. The simplest plan is to have Tables giving the L sines for each second, for the first few degrees of the quadrant.

II. In the following method we require a Table of the same size as that in method I., but it is a Table in which the differences are *insensible*. Accordingly we can with this table calculate the log sine of an angle which lies between two consecutive seconds. The method is as follows:

Let θ be the circular measure of n seconds. Then when θ is small $\theta = n \,.\, \sin 1''$ very nearly. Hence

$$\log \frac{\sin \theta}{\theta} = \log \frac{\sin n''}{n \sin 1''} = \log \sin n'' - \log n - \log \sin 1'',$$

$$\therefore\ L \sin n'' = \log \frac{\sin \theta}{\theta} + L \sin 1'' + \log n$$

$$= \log n + \left(\log \frac{\sin \theta}{\theta} + L \sin 1''\right).$$

Hence, if a table is constructed giving the values of

$$\left(\log \frac{\sin \theta}{\theta} + L \sin 1''\right)$$

for every second, for the first few degrees of the quadrant, we can, when the angle is known, find the value of

$$\log \frac{\sin \theta}{\theta} + L \sin 1''$$

from this table, while the value of $\log n$ can be found from the ordinary Table of the logs of numbers. And hence $L \sin n''$ can be found.

Moreover $\frac{\sin\theta}{\theta} = 1 - \frac{1}{6}\theta^2 +$ higher powers of θ, and θ is by hypothesis the circular measure of an angle less than 5°, so that $\frac{\sin\theta}{\theta} = 1 - \frac{1}{6}\theta^2$ approximately. Hence the differences in $\log\frac{\sin\theta}{\theta}$ when θ is small, will be insensible (*i. e.* will not affect the seventh place of decimals in the result).

[For, $\log\frac{\sin(\theta+\delta)}{\theta+\delta} - \log\frac{\sin\theta}{\theta} = \log\{1 - \frac{1}{6}(\theta+\delta)^2\} - \log(1 - \frac{1}{6}\theta^2)$ the largest term in which, is $-\frac{1}{3}\delta\theta$, *i.e.* the product of two small quantities.]

Therefore we shall not introduce any sensible error in a result obtained from the formula

$$L\sin n'' = \log n + \left(\log\frac{\sin\theta}{\theta} + L\sin 1''\right)$$

if we take the nearest value of $\log\frac{\sin\theta}{\theta} + L\sin 1''$ in the Table. The ordinary table gives the value of $\log n$. Hence we can find $L\sin n''$, even when n is not a whole number.

III. **Maskelyne's Method.** [This method is used in the absence of the special Tables required in I. and II.]

When θ is small, we have

$$\sin\theta = \theta(1 - \tfrac{1}{6}\theta^2)\,; \quad \cos\theta = 1 - \tfrac{1}{2}\theta^2\,;$$

$$\therefore \quad \frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{6} = (1 - \tfrac{1}{2}\theta^2)^{\frac{1}{3}} \text{ approximately,}$$

$$= (\cos\theta)^{\frac{1}{3}}, \text{ neglecting higher powers of } \theta \text{ than } \theta^3.$$

Hence $$\log\sin\theta = \log\theta + \tfrac{1}{3}\log\cos\theta.$$

Now, when θ is small the differences of $\log\cos\theta$ are insensible (Art. 112), and if θ be given we can therefore find $\log\sin\theta$ at once.

If we are given $\log \sin \theta$, we must first find from the table the nearest value of θ, and thence find a value of $\log \cos \theta$ which will not *sensibly* differ from the exact value, and then we get

$$\log \theta = \log \sin \theta - \tfrac{1}{3} \log \cos \theta,$$

and we thus get a second approximation for the value of θ.

With the Table of L tangents we proceed thus.

$$\begin{aligned}\log \tan \theta &= \log \sin \theta - \log \cos \theta \\ &= \log \theta - \tfrac{2}{3} \log \cos \theta \text{ approximately from above,}\end{aligned}$$

and this result may be used in a similar manner.

Example i. *Find* L *sin* $1°30'27''·2$.

Let x seconds $= \theta$ radians.

Then $$\pi x = 180 \times 60 \times 60\, \theta,$$

$$\therefore \log \theta = \log x + \bar{6}·6855749.$$

Here $1°30'27''·2 = 5427·2$ seconds;

therefore $$x = 5427·2.$$

$$\therefore 10 + \log \theta = \log 5427·2 + 4·6855749;$$

$$\begin{aligned}\therefore L \sin \theta &= 3·7345758 + 4·6855749 - \tfrac{1}{3}(·0001504) \\ &= 8·4201006.\end{aligned}$$

Example ii. *Find* θ *when* L *sin* $\theta = 8·1021832$.

From the tables by the ordinary rule we find $\theta = 43'30''$.

Hence if x be the number of seconds in θ

$$10 + \log \theta = \log x + 4·6855749 = L \sin \theta + \tfrac{1}{3}(L \sec \theta - 10);$$

$$\begin{aligned}\therefore \log x &= 8·102832 + \bar{5}·3144251 + \tfrac{1}{3}(·0000348) \\ &= 3·4166257 = \log 2609·88;\end{aligned}$$

$$\therefore \theta = 43'29''·88.$$

The student should notice the equation

$$10 + \log \theta = \log x + 4·6855749,$$

where x seconds $= \theta$ radians.

EXAMPLES. XXXIII.

(1) Find the following Tabular Logs.

(i) $L \sin 1^\circ 44' 36''\cdot 8$. (ii) $L \sin 39' 8''\cdot 4$. (iii) $L \tan 1^\circ 44' 36''\cdot 8$.

(2) Find the angle θ from the following equations:

(i) $L \sin \theta = 8\cdot 4832462$. (ii) $L \sin \theta = 8\cdot 2089620$.

(iii) $L \tan \theta = 8\cdot 4834473$.

(3) Prove that, if n be the number of seconds in an angle θ

$$L \tan \theta = \log n + 4\cdot 6855749 + \tfrac{2}{3}\,(L \sec \theta - 10).$$

114. In practical work it is always advisable to avoid as much as possible that part of a Table in which the differences are insensible. For example, a slight error in the calculation of the sine of an angle nearly 90° would entail a large error in the derived magnitude of the angle. This point is of such great practical importance that we have treated it at some length in the next chapter.

115. The preceding articles afford examples of an important general principle which is of great use in higher mathematics.

If a continuous function of a variable x increases as x approaches a certain value a, and begins to diminish directly x has passed the value a, then the ratio of the differences of the function to the corresponding small differences in the variable x will diminish and approach to zero as a limi when x approaches a.

Thus, $\sin \theta$ is a continuous function of θ which incr as θ approaches $\frac{1}{2}\pi$, and it begins to diminish directl passed through the value $\frac{1}{2}\pi$; hence, as is proved in A the ratio of $\sin(\theta + \delta) - \sin \theta$ to δ tends to become as θ approaches $\frac{1}{2}\pi$.

116. To sum up the Results of this Chapter.

The Rule of Proportional Differences may be used without sensible error in the following cases:

I. For a Table of Common Logarithms giving the logs of all numbers from 10000 to 100000.

II. For a Table of Trigonometrical Ratios calculated for intervals of one minute from 0° to 90°.

Except in the case of

the tangent and secant of angles greater than 45°,

the cotangent and cosecant of angles less than 45°.

III. For a Table of the Tabular Logarithms of Trigonometrical Ratios calculated for intervals of $10''$ from 0° to 90°.

Except in the case of

the L sines and L cosecs of angles less than 5°,

the L cosine and L secants of angles greater than 85°,

the L tans and L cotans of angles less than 5° and greater than 85°.

117. The results of this Chapter may also be obtained without actual reference to the *expansions* of $\sin\theta$ and $\cos\theta$ in terms of θ, by the aid of the fact that the difference between $\sin\delta$ and δ is less than $\frac{1}{6}\delta^3$; so that when δ is less than 1 degree, $\sin\delta$ and δ differ by less than $\cdot 0000008$.

The method of procedure is suggested in the following nple.

mple. Prove that $\sin(\theta+\delta)-\sin\theta=\sin\delta\cos\theta-2\sin\theta\sin^2\frac{1}{2}\delta$; *o that as far as* seven *places of decimals* $\sin(\theta+\delta)-\sin\theta=\delta\cos\theta$.

$$\sin(\theta+\delta)-\sin\theta=\sin\delta\cos\theta-\sin\theta(1-\cos\delta)$$
$$=\sin\delta\cos\theta-2\sin\theta\sin^2\tfrac{1}{2}\delta.$$

Now $\sin\delta$ differs from δ by less than $\frac{1}{6}\delta^3$, and $\sin^2\frac{1}{2}\delta$ differs from $\frac{1}{4}\delta^2$ by less than this. Hence the possible error when we take $\delta\cos\theta$ for the right-hand side is less than $\frac{1}{2}\delta^2$; that is, less than $\cdot 0000001$, since δ is less than the circular measure of $1'$.

EXAMPLES. XXXIV.

[In these examples δ is the circular measure of any angle less than $1'$.]

(1) Prove that $\cos(\theta-\delta)-\cos\theta=\sin\delta\sin\theta\,(1-\cot\theta\tan\frac{1}{2}\delta)$.

Hence prove that as far as seven places of decimals

$$\cos(\theta-\delta)-\cos\theta=\delta\sin\theta$$

(2) Prove that

$$\tan(\theta+\delta)-\tan\theta=\tan\delta\sec^2\theta\left(\frac{1}{1-\tan\theta\tan\delta}\right)$$

Hence obtain the results of Arts. 103, 104.

(3) Prove that $\cot(\theta-\delta)-\cot\theta=\delta\operatorname{cosec}^2\theta$ approximately.

(4) Prove that

$$\sec(\theta+\delta)-\sec\theta=\frac{\tan\delta\sin\theta\,(1+\tan\frac{1}{2}\delta\cot\theta)}{\cos^2\theta\,(1-\tan\theta\tan\delta)}:$$

hence prove, except when θ is small or nearly equal to $\frac{1}{2}\pi$, that

$$\sec(\theta+\delta)-\sec\theta=\delta\sin\theta\sec^2\theta$$

(5) Prove as in example (4) that

$$\operatorname{cosec}(\theta-\delta)-\operatorname{cosec}\theta=\delta\cos\theta\operatorname{cosec}^2\theta.$$

CHAPTER IX.

ERRORS IN PRACTICAL WORK.

118. We have already [E. 217, 227] called the student's attention to the approximate nature of all observed measurements. [See also the Author's Arithmetic, Chap. VIII.]

EXAMPLE. Let the student take any well-defined length, say of 6 or 7 inches, and attempt to ascertain its measure, say to the hundredth part of an inch; and let him repeat the process at another time with different instruments. He will find that unless he makes his measurements with the utmost care, and unless his instruments are very accurately constructed, his two results will in all probability be different.

Such an observation as the above even when made with the greatest care can only be taken as *correct to three significant figures.* If the measurement has to be made correct to a thousandth part of an inch or to any higher degree of accuracy, the student will easily understand that it will be necessary to employ specially constructed instruments. The ordinary diagonal scale or vernier cannot be read with accuracy to the thousandth part of an inch.

119. The student must carefully distinguish between mistakes and errors. By taking sufficient trouble a *calculation* can always be made to attain any required degree of accuracy; so that in what follows we are not concerned with mistakes or inaccuracies in calculation at all.

120. An **error** may be defined as follows.

Suppose an observation made and the result known to be accurate as far as a certain number of significant figures, according to the degree of approximation thought necessary or possible, under the circumstances.

The measure taken may possibly give the magnitude of the quantity with *absolute* accuracy, we cannot say whether it does or not. What we do know is that the difference between the actual magnitude of the quantity and the assumed magnitude is *less than a certain quantity.* This quantity is the *possible error;* and it should be so small that it is either considered of no importance, or is beyond the limit of observation in the circumstances of the case.

121. It is clearly not necessary to carry our calculations to any higher degree of approximation than that represented by the assumed measure.

122. In the practical application of Trigonometry to Engineering and Land Surveying we are concerned with two different kinds of measurements. (1) The measurement of **lines.** (2) The measurement of **angles.** The measurement of a line of any length with anything like the accuracy of five or six significant figures is a very difficult and tedious operation, and is but rarely performed. We know that by the methods of Trigonometry the known length of one line may be made the basis of the calculation of the lengths of all other lines in the survey of a country.

123. The **importance** of an **error in linear measurement** is generally measured by the ratio of the error to the estimated length of the distance under consideration.

Example. The problem of calculating the distance of the Sun's centre from that of the earth is beset with such great practical difficulties, that astronomers are only able to say that it is *about* 92 millions of miles. If we knew the distance to *within* a hundred thousand miles, that is, to within about a thousandth part of the distance, we should consider the distance to be known with wonderful accuracy. In a distance of this magnitude an error of a few thousands of miles is of *no importance.*

The **importance** of an **error in angular measurement** depends in general simply on the magnitude of the error.

124. If the measure of any **length** is known accurately to seven figures it is **practically exact.** In other words it is known to within the limits of observation.

EXAMPLE. A base line on Salisbury Plain measured with extreme care for the purposes of the Ordnance Survey in England is about 36578 feet in length, and the error is considered to be certainly less than 2 or 3 inches. That is, the error is less than a hundred thousandth part of the whole, and the measurement has been made correct to six significant figures.

The greatest accuracy possible in the measurement of **angles** is attained when the error is known to be not much greater than the tenth part of a second. The tenth part of a second is about the two millionth part of a radian. This degree of accuracy is only attainable under special conditions and with the largest and best instruments.

125. It sometimes happens in the course of a calculation that an error *rises* in importance in consequence of its being multiplied by a very large number.

We may illustrate this by an example.

The height h of a tower is ascertained by measuring a horizontal line a from its base and observing the angle of elevation θ of the top of the tower from the end of that line.

Then we obtain $\qquad h = a \tan \theta.$

Now supposing that we are liable to an error not greater than δ in the observed magnitude of θ *we require to know how this will affect the accuracy of the calculated height h.*

We know that the error in θ does not exceed δ. Hence we know that the consequent error in h cannot exceed k

where $\qquad h + k = a \tan (\theta + \delta)$

Hence $$k = a\{\tan(\theta + \delta) - \tan\theta\}$$
$$= a\delta\sec^2\theta$$ neglecting squares and higher powers of δ. [Art. 103]

Hence the ratio of the error k to the calculated height h is

$$\frac{\delta}{\sin\theta\cos\theta} = \frac{2\delta}{\sin 2\theta}.$$

126. The above result is very instructive.

Suppose the measurements are made with the greatest possible care, so that δ is beyond the limits of observation and may be neglected.

Then we see that in general the importance of the possible 'error' in the calculated height, *i.e.* $2\delta \operatorname{cosec} 2\theta$ is, in general, comparable with δ, and is therefore very small. Also this error is least important when $\operatorname{cosec} 2\theta$ is least, *i.e.* when θ is $\frac{1}{4}\pi$

There are two cases however when the error may become of sufficient importance to render the result practically inaccurate. I. when θ is small, II. when θ is nearly $\frac{1}{2}\pi$.

In the first case the error itself is not large, but it is large *compared with the height to be measured.*

In the second case the error itself is very large, and although the height to be measured is large compared with the base a, the importance of the error is also large compared with δ.

It is a difficulty of this latter kind which renders the estimated distance of the sun from the earth so untrustworthy.

127. We have seen that the ratio of the difference in the sine of an angle to the difference in the angle is small when the angle is nearly $\frac{1}{2}\pi$. That is to say, to a small error in the sine would correspond a large error in the angle.

Now, if the sine of an angle has been calculated from observations, and it is found that the value of the sine is nearly unity, we could not without risk of a large error use the value of the angle obtained from the Tables. For, our *observations are known to be liable to errors* (whose magnitude depends on the instruments used, etc.), and therefore the calculated value of the sine under consideration is liable to an error of the same kind. Consequently the calculated value of the angle would be liable to a much *larger error.* And this larger error would possibly affect all results in which the magnitude of the angle was used.

Accordingly in practical work an observer would when possible arrange his measurements so as to avoid such a difficulty—in the working out of a problem—as the necessity for obtaining from the value of its sine, the magnitude of an angle nearly equal to a right angle.

128. The method of Art. 125, which may be applied generally, is of very great importance in practical work; for an observer can often in this way discover beforehand whether any proposed arrangement of his measurements is defective and likely to give unreliable results.

Thus; if the measure of a distance or of any trigonometrical function of an angle be found by means of observed angles and distances, the result is expressed by some formula containing the Trigonometrical functions of the observed angles. If a small error δ be known or suspected in an observed angle θ, we can find the consequent error in the calculated distance by expanding this formula by the methods of the last chapter in ascending powers of δ. Then, δ being so small as to be detected with difficulty, δ^2 and higher powers of δ must be quite beyond the limits of observation. We can in this way estimate the *importance* of a small error in observation.

Example I. A vertical pole a feet high stands on the top of a cliff, and from a point on the shore the angles of elevation α and β of the top and bottom of this pole are observed. The height of the cliff h is given by
$$h = a\frac{\sin\beta\cos\alpha}{\sin(\alpha-\beta)}. \qquad \text{[E. LXXII. (9).]}$$

Now suppose an error δ to have occurred in the observed measurement of the angle α, *required the consequent error h' in the calculated height of the cliff.*

We have,
$$h + h' = a\frac{\sin\beta\cos(\alpha+\delta)}{\sin(\alpha-\beta+\delta)}$$
$$= \frac{a\sin\beta\{\cos\alpha - \delta\sin\alpha - \text{etc}\}}{\sin(\alpha-\beta)+\delta\cos(\alpha-\beta)-\text{etc}}$$
$$= \frac{a\sin\beta\cos\alpha}{\sin(\alpha-\beta)}\frac{\{1-\delta\tan\alpha\}}{\{1-\delta\cot(\alpha-\beta)\}} \text{ neglecting } \delta^2 \text{ etc}$$
$$= \frac{a\sin\beta\cos\alpha}{\sin(\alpha-\beta)}\{1-\delta\tan\alpha\}\{1+\delta\cot(\alpha-\beta)\},$$
therefore
$$h' = \delta\frac{a\sin\beta\cos\alpha}{\sin(\alpha-\beta)}\{\cot(\alpha-\beta)-\tan\alpha\}.$$

Thus the ratio of the error to the estimated height is
$$\delta\{\cot(\alpha-\beta)-\tan\alpha\}.$$

Example II. The centre O of the circumscribing circle (radius R) of a triangle ABC is found by measuring the angles CBO and BCO, each equal to $(\frac{1}{2}\pi - A)$; prove that, if the same small error θ is made in the measurement of each of these angles, *the consequent error in the length of OA is* $\theta \,.\, R \,.\, \operatorname{cosec} A \cos(B-C)$.

Let O be the centre, O' its observed position, then
$$OA = R = OB = OC.$$

Let $O'A = R - x$, so that the required error in OA is x; then neglecting θ^2 etc.
$$O'C = \frac{a}{2\sin(A-\theta)} = \frac{a}{2\sin A - 2\theta\cos A} = \frac{a}{2\sin A} + \theta\frac{a\cot A}{2\sin A}$$
$$= R + R\theta\cot A;$$
also
$$(R-x)^2 = O'C^2 + b^2 - 2O'Cb\cos OCA$$
$$= R^2 + 2R^2\theta\cot A + b^2 - 2Rb(1+\theta\cot A)\sin(B-\theta),$$
$$\therefore \quad -2Rx = 2R^2\theta\cot A + b^2 - R2b\sin B$$
$$-2Rb\theta(\cot A\sin B + \cos B)$$
$$= 2R^2\theta\cot A - \theta\, 4R^2\sin B\sin(A+B)\operatorname{cosec} A;$$
$$\therefore \quad x = R\theta\operatorname{cosec} A\{-\cos A + 2\sin B\sin C\}$$
$$= R\theta\operatorname{cosec} A\cos(B-C). \qquad \text{Q. E. D.}$$

EXAMPLES. XXXV.

(1) A triangle is solved from the given parts A, b, c, if there is a small error δ in the angle A prove that the consequent error in the calculated area of the triangle ABC is approximately

$$\tfrac{1}{2}\delta bc \cos A.$$

(2) A triangle is solved from the given parts A, b, c, if there is a small error δ radians in A, prove that the consequent error in B is

$$-\delta \sin B \cos C \operatorname{cosec} A \text{ radians.}$$

(3) The lengths of two sides of a triangle are known, and the included angle A is measured; prove that if there is a slight error α in the measurement of A, the corresponding error in the length of the perpendicular from A on BC will be

$$-\frac{bc \cos B \cos C}{a}\alpha.$$

(4) If the sides of a triangle be measured and a small error c' exist in the measured value of c, prove that the consequent error in the diameter of the circumscribing circle is

$$\frac{c' \cos A \cos B}{\sin A \sin B \sin C}.$$

(5) The height and distance of an inaccessible object are found by observing the angles of elevation α and β at two points A and B in a horizontal line through the base of the object, the distance between A and B being known; if the same error be made in each in consequence of an imperfect observation of the horizontal, show that the ratio of the error in the calculated height of the object to that in the calculated distance is

$$\tan(\alpha+\beta) : 1.$$

(6) The area of a quadrilateral $AOBQ$ right-angled at A and B is to be determined from observations of the angle AOB, and the lengths (a and b) of OA and OB. Prove that the area is

$$\tfrac{1}{2}\{2ab - (a^2+b^2)\cos\theta\} \operatorname{cosec}\theta,$$

and that if a small error δ be made in the observation of the angle AOB the consequent error in the area is

$$\tfrac{1}{2}\delta \,.\, AB^2 \,.\, \operatorname{cosec}^2 AOB.$$

(7) It is observed that the altitude of the top of a mountain at each of the points A, B and C where ABC is a horizontal triangle is α. Shew that the height of the mountain is

$$\tfrac{1}{2} a \tan \alpha \operatorname{cosec} A.$$

If there be a small error n'' in the altitude at C the true height is very nearly

$$\tfrac{1}{2} \cdot \frac{a \tan \alpha}{\sin A} \left\{ 1 + \frac{\cos C}{\sin A \sin B} \cdot \frac{\sin n''}{\sin 2\alpha} \right\}.$$

(8) If in a triangle ABC the observed lengths of a, b, c are 5, 4, 6 and there is known to be a small error in the measurement of c, determine which angle can be found from the formula

$$\tan \frac{A}{2} = \sqrt{\left\{ \frac{(s-b)(s-c)}{s(s-a)} \right\}}$$

with the greatest accuracy. [Result. B]

CHAPTER X

EXAMPLES OF THE APPLICATION OF TRIGONOMETRY TO GEOMETRICAL PROBLEMS.

129. In this chapter we shall use the following notation.

D, E, F are the feet of the perpendiculars drawn from the angular points A, B, C of the triangle ABC to the opposite sides.

AD, BE, CF intersect in a point P which is called the **orthocentre** of the triangle ABC.

DEF is called a **pedal triangle** of the triangle ABC.

$A'B'C'$ are the middle points of the sides BC, CA, AB.

AA', BB', CC' intersect in a point G, which is called the **centre of gravity** of the triangle ABC.

I, I_1, I_2, I_3 are the centres of the inscribed and escribed circles of the triangle ABC; r, r_1, r_2, r_3 are their radii.

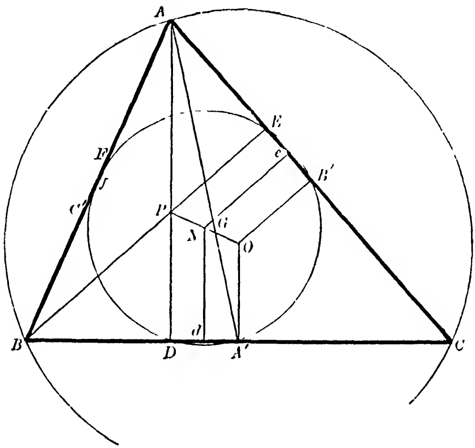
[E. 276, 278.]

O is the centre of the circumscribing circle and R its radius.

The circumscribing circle of the triangle DEF passes through $A'B'C'$ and through the middle points of each of the lines PA, PB, PC. It is called the **nine-points** *circle.* We shall denote its centre by N.

[Proofs of the propositions referred to above may be found in the appendix to Todhunter's Euclid.]

Example. To prove that P, N, G *and* O *are in one straight line, and that* PG = 2GO = 4NG, i e. *that* N *is the point of bisection, and* G *a point of trisection of* PO.



N is the centre of a circle passing through D and A'. Therefore N lies in the line bisecting DA' at right angles. This line produced bisects OP. Again, the nine-points circle passes through E and B', therefore its centre N lies on the line bisecting EB' at right angles. This line produced also bisects OP. Therefore N is the middle point of OP.

Again $\qquad AP = \dfrac{AE}{\cos PAE} = \dfrac{AE}{\sin C} = \dfrac{c \cos A}{\sin C}$,

$\therefore \qquad AP = 2R \cos A.$

But $\qquad OA' = R \cos BOA' = R \cos A$,

$\therefore \qquad AP = 2OA'.$

Hence if AA' cut PO in G, $AG : GA' = PA : OA' = 2 : 1$;

$\therefore$ $AG = 2GA'$ or G is the centre of Gravity.

Also $\qquad PG : GO = PA : OA' = 2 : 1 \qquad$ Q.E.D.

130. It is often convenient, in attempting the solution of a geometrical problem, to express the lengths of lines involved each in terms of some common unit. When the problem is one concerning a triangle, the Radius of the Circumscribing circle may be employed as the unit. Its convenience is shewn by the symmetry of the following results:

EXAMPLES. XXXVI.

Prove the following statements.

(1) $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$.

(2) $s=R(\sin A+\sin B+\sin C)=4R\cos\frac{1}{2}A\,.\cos\frac{1}{2}B\,.\cos\frac{1}{2}C$.

(3) $r=4R\sin\frac{1}{2}A\,.\sin\frac{1}{2}B\,.\sin\frac{1}{2}C$.

(4) $r_1=4R\sin\frac{1}{2}A\,.\cos\frac{1}{2}B\,.\cos\frac{1}{2}C$

(5) $AD=2R\sin B\,.\sin C$

(6) $PD=2R\cos B\,.\cos C$

(7) $AP=2R\cos A$.

(8) $OA'=R\cos A$.

(9) $S=2R^2\sin A\,.\sin B\,.\sin C$.

(10) The radius of the nine-points circle $=\frac{1}{2}R$

(11) The sides of the triangle DEF are $R\sin 2A$, $R\sin 2B$, $R\sin 2C$.

(12) The area of $DEF=\frac{1}{2}R^2\sin 2A\,.\sin 2B\,.\sin 2C$

(13) $Bd=\frac{1}{2}R(2\cos B\sin C+\sin A)$.

(14) $dN=\frac{1}{2}R\cos(B-C)$.

(15) The distances of the centres of the escribed circles from that of the inscribed circles are

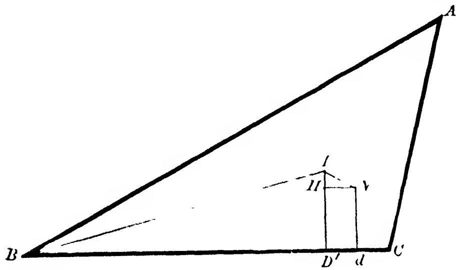
$$4R\sin\tfrac{1}{2}A,\quad 4R\sin\tfrac{1}{2}B,\quad 4R\sin\tfrac{1}{2}C.$$

(16) $AE'=R(\sin B+\sin C-\sin A)$, where E' is the point in which the inscribed circle touches AC.

Example To prove that the nine-points circle touches the inscribed circle.

We shall prove that the distance between their centres is equal to the difference of their radii.

Draw ID' perpendicular to BC and NH perpendicular to ID'.



Then (see Figure on page 131)

$$Nd = \tfrac{1}{2}(PD + OA') = \tfrac{1}{2}(2R\cos B\cos C + R\cos A)$$
$$= \tfrac{1}{2}R\cos(B - C),$$

$$\therefore \quad HI = D'I - dN = r - \tfrac{1}{2}R\cos(B - C) \qquad \text{(I)}.$$

Again $\quad Bd = \tfrac{1}{2}(BD + BA') = \tfrac{1}{2}R(2\cos B\sin C + \sin A),$

and $\quad BD' = s - b = R(\sin A - \sin B + \sin C) \quad$ [E. 280.]

$$\therefore \quad HN = \tfrac{1}{2}R(2\cos B\sin C - \sin A + 2\sin B - 2\sin C)$$
$$= R\{\sin B - \sin C - \tfrac{1}{2}\sin(B - C)\} \qquad \text{(II)}.$$

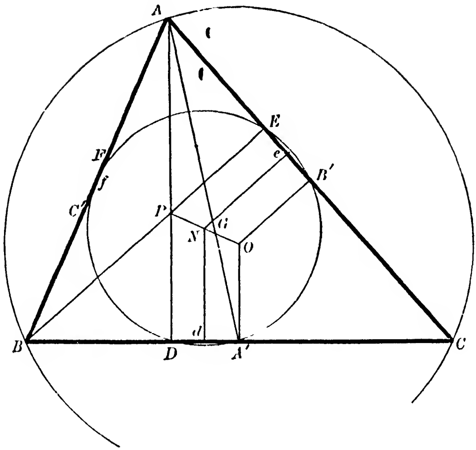
Hence $IN^2 = r^2 + (\tfrac{1}{2}R)^2 - Rr\cos(B - C) + R^2\{(\sin B - \sin C)^2 - (\sin B - \sin C)\sin(B - C)\}.$

The last bracket is equal to

$$R^2(\sin B - \sin C)\{2\cos\tfrac{1}{2}(B + C)\sin\tfrac{1}{2}(B - C) - 2\sin\tfrac{1}{2}(B - C)\cos\tfrac{1}{2}(B - C)\}$$
$$= R^2\, 4\cos\tfrac{1}{2}(B + C)\sin^2\tfrac{1}{2}(B - C)\{\cos\tfrac{1}{2}(B + C) - \cos\tfrac{1}{2}(B - C)\}$$
$$= -8R^2\sin^2\tfrac{1}{2}(B - C)\sin\tfrac{1}{2}A \,.\, \sin\tfrac{1}{2}B \,.\, \sin\tfrac{1}{2}C \qquad \text{[Ex. xxxvi. (3).]}$$
$$= -2Rr\sin^2\tfrac{1}{2}(B - C) = -Rr\{1 - \cos(B - C)\},$$

$$\therefore \quad IN^2 = r^2 + (\tfrac{1}{2}R)^2 - Rr = (r - \tfrac{1}{2}R)^2.$$

That is, (since the radius of the nine-points circle is $\tfrac{1}{2}R$) the distance between the centres of the circles equals the difference of their radii. Q.E.D.



EXAMPLES. XXXVII.

Prove the following statements:

(1) The radii of the circles circumscribing AEF, BFD, CDE are respectively $R\cos A$, $R\cos B$, $R\cos C$.

(2) $AI = 4R\sin\frac{1}{2}B \cdot \sin\frac{1}{2}C$.

(3) $AN^2 = \frac{1}{4}R^2(3 + 2\cos 2A - 2\cos 2B - 2\cos 2C)$.

$= \frac{1}{4}R^2\{1 + 8\cos A \sin B \sin C\}$.

(4) $OI^2 = R^2 - 2Rr$.

(5) $OI_1^2 = R^2 + 2Rr_1$.

(6) $OP^2 = R^2(1 - 8\cos A \cdot \cos B \cdot \cos C) = 9R^2 - a^2 - b^2 - c^2$.

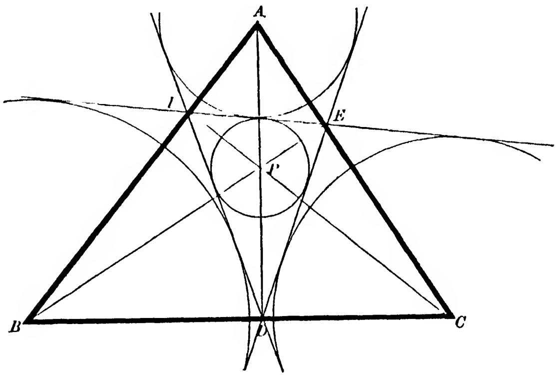
(7) $IP^2 = 4R^2(8\sin^2\frac{1}{2}A \cdot \sin^2\frac{1}{2}B \cdot \sin^2\frac{1}{2}C - \cos A \cdot \cos B \cdot \cos C)$.

(8) The area of the triangle

$IOP = -2R^2\sin\frac{1}{2}(B-C) \cdot \sin\frac{1}{2}(C-A) \cdot \sin\frac{1}{2}(A-B)$.

(9) $I_1N^2 = (\frac{1}{2}R + r_1)^2$.

131. To prove that the orthocentre is the centre of the circle inscribed in the triangle DEF.



The circle of which PC is diameter passes through E and D. Therefore the angle $EDC = EPC =$ the complement of $PCE = A$.

Similarly $FDB = FPB - EPC = A$.

Therefore $PDE = 90° - A = PDF$.

Therefore P is the centre of the circle inscribed in DEF.

Similarly A, B, C are the centres of the escribed circles of the triangle DEF.

EXAMPLES. XXXVIII.

Prove the following statements:

(1) ABC is the pedal triangle of the triangle $I_1I_2I_3$.

(2) The radius of the circle circumscribing $I_1I_2I_3$ is $2R$.

(3) DEF is the pedal triangle of the triangles APB, BPC, CPA.

(4) The radius of the circumscribing circle of the triangle $APB = R$.

(5) The circle circumscribing DEF touches the circle inscribed in ABP.

MISCELLANEOUS EXAMPLES. XXXIX.

Prove the following statements:

(1) If a new triangle is formed by joining the centres of the three escribed circles of a triangle ABC the distances of the centres of its escribed circles from the centre of its inscribed circle are

$$8R\sin\tfrac{1}{4}(B+C),\quad 8R\sin\tfrac{1}{4}(C+A),\quad 8R\sin\tfrac{1}{4}(A+B).$$

(2) The areas of the triangles $I_1I_2I_3$, I_2I_3I, I_3II_1, II_1I_2 are to one another inversely as the ratio of $r : r_1 : r_2 : r_3$.

(3) The radii of the escribed circles are the roots of the equation

$$(x^2+s^2)(x-r)=4Rx^2.$$

(4) PA, PB, PC are the roots of the equation

$$x^3-2(R+r)x^2+(r^2-4R^2+s^2)x-2R\{s^2-(r+2R)^2\}=0.$$

(5) If $p_1, p_2, p_3, -p_4$ are the perpendiculars from ABC and P on the sides of the triangle DEF, then p_1, p_2, p_3, p_4 are the roots of the equation

$$x^4-2Rx^3+\left(\frac{\Delta^2}{R^2}-2Rr'-r'^2\right)x^2-\frac{\Delta^2}{R^2}r'^2=0,$$

where r' is the radius of the circle inscribed in the triangle DEF.

(6) The area of the triangle formed by joining the points of contact of the inscribed circle is

$$4R^2\sin A\,.\sin B\,.\sin C\,.\sin\tfrac{1}{2}A\,.\sin\tfrac{1}{2}B\,.\sin\tfrac{1}{2}C.$$

(7) If the points of contact of each of the four circles touching the three sides of a triangle be joined, and the area of the triangle thus formed from the inscribed circle be subtracted from the sum of the areas of those formed from the escribed circles, the remainder will be double the area of the original triangle.

(8) If R_1, R_2, R_3 are the radii of the circles BGC, CGA, AGB, then

$$\frac{a^2(b^2-c^2)}{R_1^2}+\frac{b^2(c^2-a^2)}{R_2^2}+\frac{c^2(a^2-b^2)}{R_3^2}=0.$$

(9) If x stand for AI, y for BI, z for CI, then

$$a^4x^4+b^4y^4+c^4z^4+(a+b+c)^2x^2y^2z^2=2\{b^2c^2y^2z^2+c^2a^2z^2x^2+a^2b^2x^2y^2\}.$$

(10) If lines join the points of contact of each escribed circle of a triangle ABC with the produced sides and these lines form a new triangle, then the lines joining the corresponding vertices of the triangle are perpendicular to the sides of the former triangle and are equal to the radii of the escribed circles.

(11) Given the circumscribed and inscribed circles of a triangle, prove that the centres of the escribed circles lie on a fixed circle.

(12) The sum of the reciprocals of the perpendiculars of a triangle is equal to the sum of the reciprocals of the radii of the escribed circles.

(13) If from a point P perpendiculars PL, PM, PN are drawn to the side of the triangle ABC, prove that twice the area of the triangle $LMN = \{R^2 - (PO)^2\} \sin A \sin B \sin C$.

(14) The centres of the escribed circles must lie without the circumscribing circle, and cannot be equidistant from it unless the triangle is equilateral.

(15) $r^3 II_1 . II_2 . II_3 = IA^2 \; IB^2 . IC$

(16) The area of the triangle whose angular points are the points of contact of the inscribed circle is to the area of the triangle ABC as $r : 2R$.

(17) If DEF are the points of contact of the inscribed circle with the sides of the triangle ABC, then if AD^2, BE^2, CF^2 are in A. P., a, b, c are in H. P.

(18) From DEF perpendiculars are drawn to the adjacent sides of the triangle ABC; prove that the feet of these six perpendiculars lie on a circle whose radius is

$$R\,(\cos^2 A \cos^2 B \cos^2 C + \sin^2 A \sin^2 B \sin^2 C)^{\frac{1}{2}}.$$

(19) If on one side BC of a triangle ABC a triangle $A'BC$ is described without it such that the angles $BA'C$, CBA', $A'CB$ are equal to $\alpha\beta\gamma$ respectively, then

$$AA'^2 \sin\alpha = \sin\beta \sin\gamma \,(a^2 \cot\alpha + b^2 \cot\beta + c^2 \cot\gamma + 4\Delta).$$

(20) If a triangle be cut out in paper and doubled over so that the crease passes through the centre of the circumscribing circle and one of the angles A, the area of the doubled portion is

$$\tfrac{1}{2} b^2 \sin^2 C \cos C \operatorname{cosec}(2C - B) \sec(C - B),$$

C being greater than B.

(21) If two of the angular points and the radius of the circumscribing circle of a triangle are given, the loci of the centre of the nine-point circle and of the orthocentre are circles.

(22) Prove that a triangle can be constructed whose sides are $a\cos A$, $b\cos B$, $c\cos C$ and that its area is

$$2\Delta \,.\, \cos A \,.\, \cos B \,.\, \cos C.$$

(23) If R_1, R_2, R_3 are the radii of the circumscribed circles of BIC, CIA, AIB, prove that $R_1^2 \,.\, R_2^2 \,.\, R_3^2 = R^3 \,.\, AI \,.\, BI \,.\, CI$.

(24) If the two straight lines which bisect the angles A and C of a triangle ABC, meet the circumference of the circumscribing circle in R and S, then RS is divided by CB, BA into three parts which are in the ratio

$$\sin^2 \tfrac{1}{2}A : 2\sin\tfrac{1}{2}A \,.\, \sin\tfrac{1}{2}B \,.\, \sin\tfrac{1}{2}C \,.\, \sin^2\tfrac{1}{2}C.$$

(25) If a point be taken in an equilateral triangle such that its distances from the angular points are proportional to the sides of a triangle ABC, the angles between these distances will be

$$\tfrac{1}{3}\pi + A,\ \tfrac{1}{3}\pi + B,\ \tfrac{1}{3}\pi + C$$

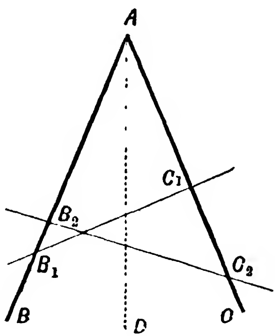
(26) $\tan IOI_1 = \pm \dfrac{2(\sin B - \sin C)}{2\cos A - 1}$.

CHAPTER X a.

On Antiparallels.

132. It is usual to use the abbreviations, *circumcentre, incentre, gravity centre* etc. for *the centre of the circumscribing circle*, etc.

DEF. Let BAC be any angle and let AD be the bisector of the angle; then the line B_1C_1 is said to be **antiparallel** to the line B_2C_2 with respect to the angle A when B_1C_1 and B_2C_2 are equally inclined to AD and are not parallel.



When two antiparallels with respect to an angle pass through the angular point they are called **Isogonal Conjugates.**

EXAMPLES. XL.

(1) Lines which are antiparallel to the same straight line with respect to the same angle are parallel to each other.

(2) If B_1C_1, B_2C_2 are antiparallels with respect to the angle BAC, prove that the angles which B_1C_1 makes with AB are equal to the angles which B_2C_2 makes with AC.

(3) With the notation of Question (2), prove that a circle goes round B_1, C_2, C_1, B_2.

(4) Lines which are perpendicular to two antiparallels respectively are antiparallel to each other with respect to the same angle.

(5) With the notation of Question (2), if AG_1 bisects B_1C_1 and AG_2 bisect B_2C_2, prove that AG_1 is an antiparallel to AG_2 with respect to the angle A.

(6) If AX, AY are (isogonal conjugates or) antiparallels with respect to BAC and XB_1, XC_1, YB_2, YC_2 are drawn perpendicular to AB and AC respectively, prove (i) that $XB_1 . YB_2 = XC_1 . YC_2$, (ii) that the points $B_1B_2C_1C_2$ are concyclic, (iii) that B_1C_1 is perpendicular to AY.

(7) Prove that the tangent at A to the circumcircle of a triangle ABC is antiparallel to the side BC with respect to A.

(8) The lines joining the feet of the perpendiculars of a triangle are antiparallel to the sides with respect to the opposite angles respectively.

ANTICENTRES.

133. PROP. *If a point* O *be taken in the plane of a triangle* ABC *and* OA, OB, OC *are drawn, then the antiparallels to* OA, OB, OC *passing through* A, B, C *respectively, with respect to the angles* A, B, C *respectively meet in a point.*

OR. *If three lines drawn from the angular points* A, B, C *of a triangle meet in a point, their isogonal conjugates with respect to* A, B, C *respectively also meet in a point.*

Let AO' be the antiparallel through A of AO with respect to A.

And let O' be any point in AO'.

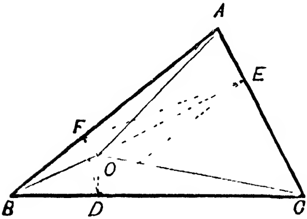
Draw OD, OE, OF, $O'D'$, $O'E'$, $O'F'$ perpendiculars to BC, CA, AB, BC, CA and AB respectively.

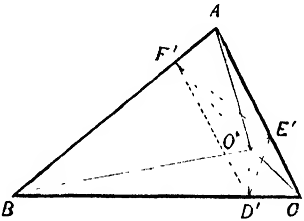
Then a circle goes round $FOEA$, and round $F'O'E'A$, therefore $\quad FOE = 180° - A = F'O'E'$

Also $OEF = OAF = O'AE'$ [because $O'A$ is the antiparallel of OA] $= O'F'E'$,

therefore the triangles FOE and $E'O'F'$ are similar

So that $\quad OE \,.\, O'E' = OF \,.\, O'F'$





Again if BO' be the antiparallel through B of BO with respect to B, we may prove that

$$OF \,.\, O'F' = OD \,.\, O'D'.$$

Therefore $\quad OD \,.\, O'D' = OE \,.\, O'E'.$

And the angle $\quad D'O'E' = DOE.$

Therefore the triangles $D'O'E'$ and EOD are similar

Therefore the angle

$$OCD = OED = O'D'E' = O'CE'.$$

That is, $O'C$ is an antiparallel of OC with respect to C, which proves the proposition.

134. DEF. The point O' is called the **Anticentre** of O with respect to the triangle ABC.

EXAMPLES. XLI.

(1) The bisectors, drawn from the angular points, of the antiparallels to the sides of a triangle, with respect to the opposite angles respectively, meet in a point. This point is called the **Cosine Centre.**

(2) The perpendiculars, drawn from the angular points, to the antiparallels to the sides of a triangle with respect to the opposite angles respectively, meet in a point.

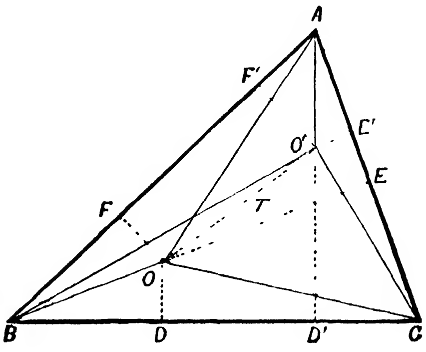
(3) Shew that the circumcentre is the anticentre of the orthocentre

(4) Shew that the point in Question (1) is the anticentre of the gravity centre.

(5) Prove that the Incentre is its own anticentre

(6) If the isogonal conjugates AO, AO' meet the side BC in M and N and the circumcircle in $M'N'$, then the rectangles $AB \cdot AC$, $AM \cdot AN'$, $AM' \cdot AN$ are equal

135. PROP. *If* D, E, F, D′, E′, F′ *are the feet of the perpendiculars on the sides of the triangle* ABC *from any point* O *and its anticentre* O′ *with respect to* ABC, *then* D, E, F, D′, E′, F′ *are on the circumference of a circle whose centre bisects* OO′.



Let T be the middle point of OO'.

Then $TD^2 = TD'^2 = TO^2 + OD \cdot O'D'$ †

$$= TO^2 + OE \cdot O'E' \qquad \text{[Art. 132]}$$
$$= TE^2$$
$$= TO^2 + OF \cdot O'F'$$
$$= TF^2$$

EXAMPLES. XLII.

(1) Prove that the centre of the nine-points circle bisects the line joining the orthocentre and the circumcentre.

(2) If in the figure of Art 135, P be the point in BC such that the angles OPB, $O'PC$ are equal, then $OP + O'P = 2TD$.

(3) With the construction of (2), prove that if Q be the point in AC such that $OQC = O'QA$, then the angles COP, COQ are equal.

(4) O, O' are foci of an ellipse inscribed in the triangle ABC.

136. In a triangle ABC there are three systems of antiparallels which arise from the sides and angles;

for there is a system of antiparallels to each of the three sides with respect to the opposite angle.

The antiparallels to BC with respect to A are parallel to the tangent to the circumcircle at A.

Hence the above three systems of antiparallels are the systems of perpendiculars to OA, OB, OC respectively, where O is the circumcentre.

† Draw Od' parallel to DD' cutting $O'D'$ in d', draw Td perpendicular to DD'.

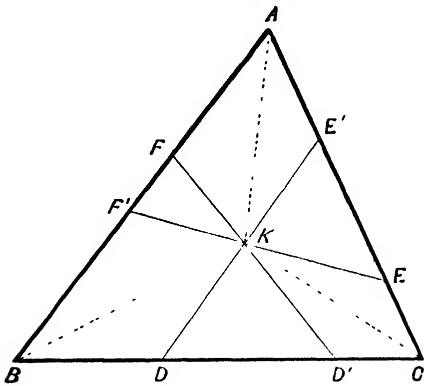
Then $TD^2 = Td^2 + Dd^2 = Td^2 + \frac{1}{4} DD'^2$

$$= \tfrac{1}{4}(OD + O'D')^2 + \tfrac{1}{4}\{OO'^2 - O'd'^2\}$$
$$= \tfrac{1}{4}(OD + O'D')^2 + OT^2 - \tfrac{1}{4}(O'D' - OD)^2$$
$$= OT^2 + OD \cdot O'D'.$$

THE COSINE CIRCLE.

137. PROP. *The bisectors, drawn from the angular points of a triangle, of the antiparallels to the sides with respect to the opposite angles, meet in a point.*

[This follows at once from the proposition of Art. 133, but we give an independent investigation]



Take *any* point K and draw through K, EKF', FKD', DKE' antiparallels to BC, CA, AB with respect to A, B, C respectively.

Then since EKF' is antiparallel to BC with respect to A, therefore the angle $KF'A = C$.

Similarly, since FD' is antiparallel to AC with respect to B, therefore the angle $KFB = C$.

Therefore $KF'F = KFF'$ and $FK = F'K$.

Similarly $DK = D'K$, and $EK = E'K$.

Now, let K be the point of intersection of the bisectors through B and C, of the series of antiparallels to AC, AB.

Then $KD' = KF = KF'$ and $KD = KE' = KE$,

but $KD = KD'$, therefore $KE = KF'$.

That is, AK is the bisector of the antiparallels to BC with respect to A. Q. E. D.

138. The circle with centre K and radius KE passes through E, E', F, F', D, D'.

This circle is called the **Cosine Circle.**

The point K is called the **Cosine Centre.**

139. Let ρ be the radius of the cosine circle, then

$$\begin{aligned} DD' - 2\rho \cos KDD' &= 2\rho \cos A, \\ EE' &= 2\rho \cos B, \\ FF' &= 2\rho \cos C. \end{aligned}$$

Hence the name, *cosine* circle.

140. Let the perpendiculars from K on the side be indicated by x, y, z, then

$$x = \rho \sin A, \quad y = \rho \sin B, \quad z = \rho \sin C;$$

$$\therefore \quad x = \rho \cdot \frac{a}{2R}, \quad y = \rho \cdot \frac{b}{2R}, \quad z = \rho \cdot \frac{c}{2R}.$$

Now $xa + yb + zc = 2$ area of triangle $ABC = 2\Delta$;

$$\therefore \frac{2\Delta}{a^2 + b^2 + c^2} = \frac{xa + yb + zc}{a^2 + b^2 + c^2} = \frac{\rho}{2R} \cdot \frac{a^2 + b^2 + c^2}{a^2 + b^2 + c^2} = \frac{\rho}{2R} = \frac{x}{a}.$$

Now

$$\begin{aligned} \frac{a^2 + b^2 + c^2}{4\Delta} &= \frac{2bc \cos A + 2ca \cos B + 2ab \cos C}{4\Delta} \\ &= \cot A + \cot B + \cot C; \end{aligned}$$

$$\therefore \quad \frac{a}{2x} = \frac{b}{2y} = \frac{c}{2z} = \cot A + \cot B + \cot C.$$

EXAMPLES. XLIII.

(1) Prove that AK passes through K_a the intersection of tangents at B, C to the circumcircle.

(2) If the tangents to the circumcircle at BC of a triangle intersect at K_a, prove that the angles K_aAI and GAI are equal.

(3) Prove that BK_a is antiparallel to BC with respect to A.

(4) Prove that the line drawn through K_a parallel to the tangent at A to the circumcircle to cut the sides AB, AC produced, is bisected at K_a.

(5) AK divides the parallels to the sides BC in the ratio $c^2 : b^2$.

(6) AK bisects the line joining the feet of the perpendiculars from BC on the opposite sides of the triangle ABC.

(7) If x, y, z be the perpendiculars from K on the sides of the triangle, then $$\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=\frac{2\Delta}{a^2+b^2+c^2}.$$

(8) The sum of the squares of the distances of any point Q from the sides is a minimum, when Q is at K.

(9) Prove that $\dfrac{4R}{\rho}=\cot A+\cot B+\cot C$.

THE EX-COSINE CIRCLES

141. Let the tangents to the circumcircle at A, B, C intersect in the points K_a, K_b, K_c;

K_a, K_b, K_c are called the **Ex-Cosine Centres.**

Through K_a draw $E_1K_aF_1$ parallel to K_bK_c.

Then E_1F_1 is antiparallel to BC, therefore

$$K_aE_1C=B=K_aCE_1.$$

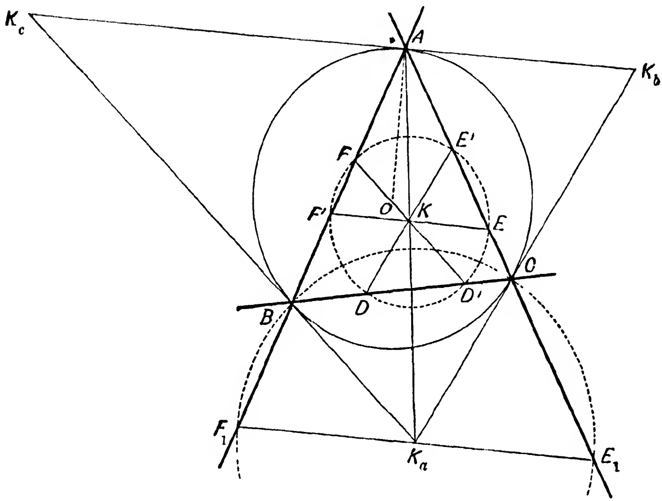
Hence $K_aE_1=K_aC$; similarly $K_aB=K_aF_1$;

but $\qquad K_aC=K_aB$; therefore $K_aE_1=K_aF_1$.

Hence the circle whose centre is K_a and radius K_aC, passes through E_1, C, B, F_1.

Now K_aC and K_aB are antiparallels with respect to C and B respectively of AB and AC, and K_a may be said to

bisect the portions into which the line K_aC is divided by the lines AC, BC.



Hence the circle centre K_a which passes through E_1, C, B, F_1 is called an **Ex-Cosine Circle.** [See Ex. (5) below.]

NOTE. AK produced bisects E_1F_1 and therefore passes through K_a

EXAMPLES. XLIV.

(1) Prove that the perpendiculars x_a, y_a, z_a from K_a on the sides BC, CA, AB of the triangle ABC are as $a:b:c$

(2) Prove that $$\frac{x_a}{a}=\frac{y_a}{b}=\frac{z_a}{c}=\frac{2\Delta}{b^2+c^2-a^2}.$$

(3) $$\frac{a}{2x_a}=\cot A.$$

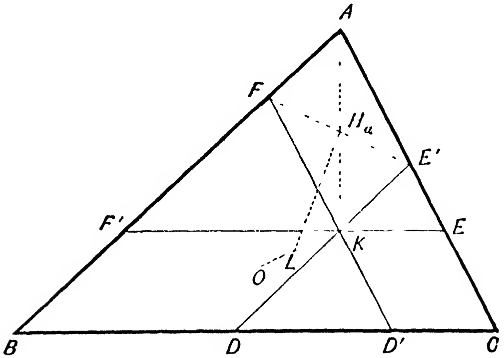
(4) If ρ_a be an ex-cosine radius, then
$$\rho_a=\frac{4R\Delta}{b^2+c^2-a^2}.$$

(5) Prove that $F_1B : BC : CE_1 = \cos C : \cos A : \cos B$,

The Lemoine Circle.

142. PROP. *The parallels to the sides of a triangle* ABC *drawn through the cosine centre K, cut the sides in six points which lie on the circumference of a circle called the* **Lemoine Circle.**

Let the parallels to the sides through K cut the sides in the points EF', FD', DE'. Let O be the circumcentre.



Bisect OK in L and let AK, FE' intersect in H_a.

Then $FKE'A$ is a parallelogram, and FE' and AK bisect each other in H_a.

Therefore FE' is antiparallel to BC with respect to A, and is therefore perpendicular to OA. [Art. 136.]

But LH_a is parallel to OA and therefore perpendicular to FE'. Also LH_a is $\frac{1}{2}OA = \frac{1}{2}R$.

And FE' is half the line through K parallel to FE'

That is, $FE' = \rho$.

Therefore $LF^2 = LH_a^2 + FH_a^2 = \frac{1}{4}(R^2 + \rho^2)$.

Similarly the squares on LE', LE, LD', LD, LF' may each be proved equal to $\frac{1}{4}(R^2 + \rho^2)$.

Therefore the six points D, E, F, D', E', F' lie on a circle, the square on whose diameter is $(R^2 + \rho^2)$. Q. E. D.

143. Draw KX, KY, KZ perpendiculars to BC, CA, AB respectively, then $x = EC \sin C$, $z = E'A \sin A$:

$$\therefore\ EC : E'A = x \sin A : z \sin C = a^2 : c^2. \qquad \text{[Art. 110.]}$$

Again, the triangle EKE' is similar to ABC

Therefore $\qquad EE' : KE' = \sin B : \sin C.$

$$\therefore\ EE' = \frac{y \sin B}{\sin A \sin C} = \frac{y \sin B}{x \sin A} . CE;$$

[since $KE' \sin A = y$, and $EC \sin C = x$]

$$\therefore\ EE' : EC = y \sin B : x \sin A = b^2 : a^2$$

Hence $\qquad CE : EE' : E'A = a^2 : b^2 : c^2.$

144. From Art. 143

$$EE' : CE + EE' + E'A = b^2 : a^2 + b^2 + c^2,$$

$$\therefore\ EE' = \frac{b^3}{a^2 + b^2 + c^2}.$$

Hence $\qquad DD' : EE' : FF' = a^3 : b^3 : c^3.$

Hence the Lemoine Circle has been called by Mr Tucker the **Triplicate Ratio Circle.**

EXAMPLES. XLV.

(1) Prove that, in the figure of Art. 142, ABC can be divided into nine triangles each similar to ABC, by joining DE', EF', FD', DF', FE', ED'.

(2) Prove that $DD' : x = a$: the perpendicular from A on BC

(3) $$DD' = \frac{xa}{b \sin C} = a^3 . \frac{\rho}{abc}.$$

(4) Let EF and $E'F'$, FD and $F'D'$, DE and $D'E'$, intersect respectively in p, q, r; prove that the triangles EpE', FpF' are similar. Hence shew that the perpendiculars from any point on Ap to the sides AB, AC respectively are in the ratio $EE' : FF' = b^3 : c^3$.

(5) Deduce from Question (4) that Ap, Bq, Cr meet in a point.

(6) The intersection P of EF' and $E'F$ is the pole of Ap with respect to Lemoine's circle; hence prove (7).

(7) If P, Q, R be the points of intersection of EF' and $E'F$, FD' and $F'D$, DE' and $D'E$, then P, Q, R are collinear.

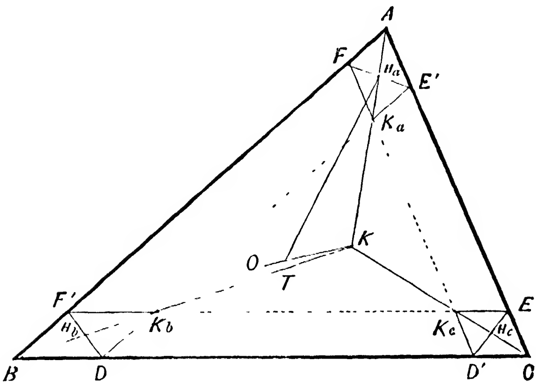
Tucker's Circles.

144. In the figure of Art. 142 let T divide OK so that

$$OT : TK = 1 : \lambda$$

Also take H_a in AK so that

$$AH_a : H_aK = 1 : \lambda.$$



Then TH_a is parallel to OA.

Through H_a draw FH_aE' antiparallel to BC;

then AK bisects FE' in H_a.

Also, OA, and therefore also TH_a, is perpendicular to FE'.

Again $TH_a : OA = KH_a : KA = \lambda : 1 + \lambda$;

$$\therefore \quad TH_a = \frac{\lambda}{1+\lambda} R$$

And $$FH_a = \frac{1}{1+\lambda} \rho; \qquad \text{[as in Art. 142]}$$

$$\therefore \quad TF^2 = TH_a^2 + FH_a^2 = \left(\frac{1}{1+\lambda}\right)^2 (\lambda^2 R^2 + \rho^2)$$

$$= TE'^2 \text{ similarly.}$$

Now with a similar construction, in KB, KC take H_b, H_c and draw the antiparallels $F'D$, $D'E$.

Then we can prove as before that

$$TF'^2 = \left(\frac{1}{1+\lambda}\right)^2 (\lambda^2 R^2 + \rho^2) = TD^2 = TD'^2 = TE^2$$

Therefore the six points D, E, F, D', E', F' lie on a circle centre T, the square on its radius being

$$\left(\frac{1}{1+\lambda}\right)^2 (\lambda^2 R^2 + \rho^2).$$

A circle thus described is called a **Tucker's Circle.**

145. Now complete the parallelograms

$$FAE', \quad F'BD, \quad D'CE.$$

Then since FE' is bisected in H_a, therefore AK_a, the diameter of the parallelogram FAE', passes through H_a and also through K; also

$$AK_a : AK = 2 : 1 + \lambda = BK_b : BK = CK_c : CK,$$

$\therefore$ K_aK_b is parallel to AB and $E'K_aK_bD$ is a straight line Similarly $F'K_bK_cE$, $D'K_cK_aF$ are straight lines

EXAMPLES. XLVI.

(1) In the figure of Art 144, prove that the triangles DEF, $D'E'F'$ are each similar to ABC.

(2) Prove the following construction for drawing a Tucker's circle. Draw KA, KB, KC, in KA take any point K_a; through K_a draw lines parallel to AB, AC cutting KB, KC in K_b and K_c.

Join K_bK_c; then the point where the lines K_aK_b, K_bK_c, K_cK_a cut the sides are points on a Tucker's circle.

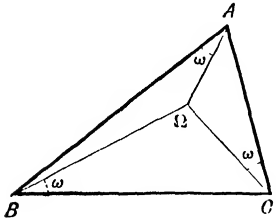
(3) The circumcentre of the hexagon $DD'EE'FF'$ bisects the distance between the circumcentres of the triangles ABC, $K_aK_bK_c$

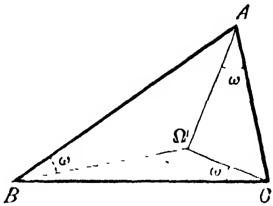
(4) Lemoine's circle is the Tucker's circle in which λ is unity.

(5) Prove that in the figure for Tucker's circle $DF' = FE' = ED'$.

The Brocard Points.

146. PROP. *To find a point* Ω *within a triangle* ABC *such that the angles* $\Omega BC = \Omega CA = \Omega AB = \omega$





Since

$$\Omega BC = \Omega CA, \quad \therefore\ B\Omega C = 180^\circ - \Omega CB - \Omega CA = 180^\circ - C.$$

Similarly $C\Omega A = 180^\circ - A$ and $A\Omega B = 180^\circ - B$

Hence $\dfrac{A\Omega}{\sin\omega} = \dfrac{b}{\sin A}$ and $\dfrac{A\Omega}{\sin(B-\omega)} = \dfrac{c}{\sin B}$.

$$\therefore\ \frac{\sin(B-\omega)}{\sin\omega} = \frac{b}{c}\cdot\frac{\sin B}{\sin A} = \frac{\sin^2 B}{\sin C \sin A};$$

$$\therefore\ \cot\omega - \cot B = \frac{\sin(A+C)}{\sin C \sin A} = \cot A + \cot C;$$

$$\therefore\ \cot\omega = \cot A + \cot B + \cot C.$$

Again, if the angles $\Omega' CB = \Omega' BA = \Omega' AC = \omega'$ we shall obtain the same equation to find ω' so that $\omega = \omega'$.

Ω' is, in fact, the anticentre of Ω with respect to ABC.

147. The points Ω and Ω' are called the **Brocard Points,** and the angle ω, the **Brocard** angle, of the triangle ABC.

Milne's construction for Ω, Ω' is

On BC, CA, AB describe circles touching CA, AB, BC respectively; these intersect in Ω

On BC, CA, AB describe circles touching AB, BC, CA respectively; these circles intersect in Ω.

148. Let ΩX, ΩY, ΩZ be perpendiculars on the sides, then it will be seen that $YXZ = B$, $XZY = A$ and $ZYX = C$, so that the triangle XYZ is similar to ABC.

Also, Ω is a Brocard point of XYZ and ω is the Brocard angle. Also since $\Omega X = \Omega B \sin\omega$, we have

$$YZ = CA \sin\omega, \quad ZX = AB \sin\omega \quad \text{and} \quad XY = BC \sin\omega\,,$$

therefore the radius of the circumcircle of $XYZ = R \sin\omega$.

Similar remarks apply to a triangle $X'Y'Z'$ formed by drawing perpendiculars to the sides from Ω'.

But by Art. 135 a circle goes round $XYZX'Y'Z'$, and

$$(\text{its radius})^2 \quad = \tfrac{1}{4}\Omega\Omega'^2 + \Omega X \,.\, \Omega' X'.$$

$$\text{Now} \qquad \Omega X = \Omega B \sin\omega = \frac{c \sin^2\omega}{\sin B} = 2R \sin^2\omega,$$

$$\text{and} \qquad \Omega' X' = \Omega' C \sin\omega = \frac{b \sin^2\omega}{\sin C}\,,$$

$$\therefore \ \Omega X \,.\, \Omega' X' = \frac{bc}{\sin B \sin C} \sin^4\omega = 4R^2 \sin^4\omega\,;$$

$$\therefore \ R^2 \sin^2\omega = \tfrac{1}{4}\Omega\Omega'^2 + \Omega X \,.\, \Omega' X' = \tfrac{1}{4}\Omega\Omega'^2 + 4R^2 \sin^4\omega\,;$$

$$\therefore \ \Omega\Omega'^2 = 4R^2 \sin^2\omega\,\{1 - 4\sin^2\omega\}.$$

EXAMPLES. XLVII.

(1) The circumcircle of the triangle $A\Omega B$ touches BC in B.

(2) Let the perpendiculars to BA through A and to AC through the mid-point of AC meet in L; let the perpendiculars to CB, through B and to BA through the mid-point of BA meet in M; let the perpendiculars to AC through C and to CB through the mid-point of CB meet in N; then the circles, centres L, M, N and radii LA, MB, NC, intersect in Ω.

(3) If x, y, z are the perpendiculars from the cosine centre on the sides of the triangle ABC, prove that $x/a = y/b = z/c = \frac{1}{2}\tan\omega$.

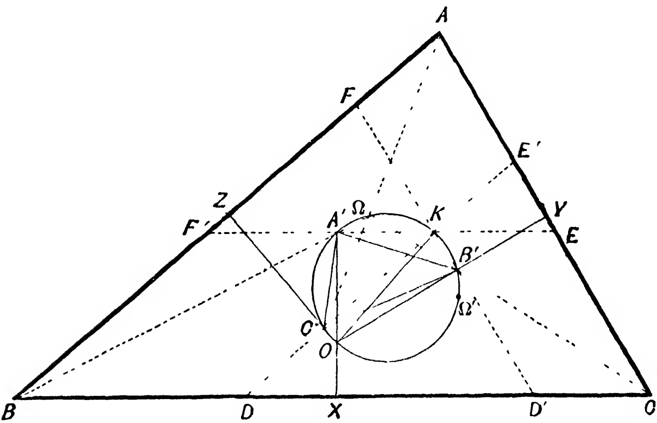
(4) $$\tan\omega = \frac{\sin A \sin B \sin C}{1 + \cos A \cos B \cos C}.$$

(5) In an equilateral triangle Ω and Ω' are coincident.

(6) In a right-angled triangle $\tan\omega = \sin A \sin B$.

THE BROCARD CIRCLE.

149. Take O the circumcentre and K the cosine-centre of the triangle ABC; the circle described upon OK as diameter is called the **Brocard Circle** of the triangle.



Through K draw EKF', FKD', DKE' parallels to the sides and through O draw XOA', YOB', ZOC' perpendiculars to the sides respectively. Then $KA'O$, $KB'O$, $KC'O$ are right angles, and A', B', C' are points on the Brocard circle.

I. The angle $B'C'A' = B'OA'$, which is equal to C; for $A'O$ and OB' are perpendicular to BC and CA respectively.

Similarly $\quad A'B'C' = B$ and $B'A'C' = A$.

Therefore the triangle $A'B'C'$ is similar to ABC.

II. Join BA' and AC' and let them intersect in Ω.

Then since $A'X =$ the perpendicular from K on BC,

$$A'X = x = \rho \sin A \quad \text{and} \quad BX = \tfrac{1}{2}a\,,$$

$$\therefore \frac{A'X}{BX} = \frac{2\rho \sin A}{a} = \frac{2\rho \sin B}{b} = \frac{B'Y}{YC} = \frac{C'Z}{ZA}.$$

That is, $\tan A'BX = \tan B'CY = \tan C'AZ$; therefore the angles $A'BX$, $B'CY$, $C'AZ$ are equal. Also

$$\Omega A'O = 180^\circ - BA'X = 180^\circ - CB'Y = 180^\circ - \Omega B'O.$$

That is, the intersection of BA' and CB' is on the Brocard circle.

Similarly AC' also cuts BA' on the Brocard circle.

Hence AC', BA', CB' are concurrent on the Brocard circle; and, since the angles $A'BX$, $B'CY$, $C'AZ$ are equal their point of concurrence, Ω, is a Brocard point of the triangle ABC.

Similarly AB', BC', CA' are concurrent and meet on the Brocard circle at the other Brocard point

III. Lemoine's circle goes round $DD'EE'FF'$, and its centre bisects OK, therefore

$$KE = F'A', \quad KF = D'B' \text{ and } KE' = DC'.$$

And $KE' = FA$, $\therefore$ $FAC'D$ is a parallelogram.

Therefore FD is parallel to $A\Omega$.

This gives a construction for finding the Brocard angle.

IV
$$\frac{DD'}{FD} = \frac{\sin DFD'}{\sin FD'D} = \frac{\sin C'AC}{\sin C} = \frac{\sin(A-\omega)}{\sin C},$$

$$\frac{F'D}{FD} = \frac{\sin DFF'}{\sin DF'A} = \frac{\sin \Omega AB}{\sin DF'B} = \frac{\sin \omega}{\sin C}.$$

$$\therefore DD' : F'D = \sin(A-\omega) \,.\, \sin\omega;$$

$\therefore$ the sides of Lemoine's hexagon are as

$$\sin(A-\omega) : \sin\omega : \sin(B-\omega) : \sin\omega : \sin(C-\omega) : \sin\omega.$$

NOTE. $\dfrac{R}{\rho} = \cot A + \cot B + \cot C = \cot\omega$; [Art. 140]

$$\therefore \rho = R\tan\omega.$$

For further information on Modern Geometry the student should consult Sequel to Euclid by the late Professor Casey of Dublin.

EXAMPLES. XLVIII.

(1) Prove that $B\Omega$ and $C\Omega'$ intersect in A'.

(2) Prove that $OK^2 = R^2 - 3\rho^2$.

(3) $\dfrac{1}{\rho} = \dfrac{s^2}{RS} - \dfrac{2}{a} - \dfrac{2}{b} - \dfrac{2}{c} = \dfrac{a^2+b^2+c^2}{abc}$.

(4) $\dfrac{1}{\rho^2} + \dfrac{1}{R^2} = \dfrac{4}{a^2} + \dfrac{4}{b^2} + \dfrac{4}{c^2}$.

(5) If AK, BK, CK produced cut the sides in DEF, prove that

(i) area of $DEF : \Delta = 2a^2b^2c^2 : (b^2+c^2)(c^2+a^2)(a^2+b^2)$.

(ii) $4AD^2 = \dfrac{2bc}{b^2+c^2}(2b^2+2c^2-a^2)$.

(iii) $4AK^2 = \dfrac{2bc}{a^2+b^2+c^2}(2b^2+2c^2-a^2)$.

(iv) $BD : CD = c^2 \quad b^2$.

(6) Lemoine Radius $= \frac{1}{2}R \sec\omega$.

(7) In a right-angled triangle the diameter of the cosine circle is the perpendicular from the right angle on the hypotenuse.

(8) The perpendicular from K on BC divides it in the ratio $a^2 - b^2 + 3c^2 : a^2 + 3b^2 - c^2$.

(9) The cosine circle cuts BC in the ratio $\cot B : \cot C : \cot A$.

(10) ω is never greater than 30°.

(11) In the fig. of Art. 142, $DF' = FE' = ED' = \dfrac{abc}{a^2+b^2+c^2}$

(12) The perimeter of Lemoine's hexagon is $\dfrac{a^3+b^3+c^3+3abc}{a^2+b^2+c^2}$ and its area is $\Delta\dfrac{a^4+b^4+c^4+b^2c^2+c^2a^2+a^2b^2}{(a^2+b^2+c^2)^2}$.

(13) In an equilateral triangle, the incircle coincides with the nine-points circle, Lemoine's circle with the cosine circle, the escribed centres with the excosine centres;

$$R = 2r = \tfrac{2}{3}r_1 = \sqrt{3}\rho = \tfrac{1}{3}\sqrt{3}\rho_a = \tfrac{1}{3}\sqrt{3}a, \quad \omega = 30°.$$

(14) If $C = 90°$, $\rho = a\sin B$, $\rho_a\rho_b = R^2$, $\rho_c = \infty$.

(15) FD is parallel to $A\Omega$; Ω, K are the Brocard points of DEF; $\Omega' K$ are the Brocard points of $D'E'F'$. [Fig. Page 154.]

CHAPTER XI

On the Use of Subsidiary Angles to facilitate Numerical Calculation.

150. In the *Elementary Trigonometry*, Art. 185, we have shewn how the Tables may be made use of in the solution of Simple Trigonometrical Equations.

It is usual to shew how the Tables may be made use of to facilitate the calculation of the roots of quadratic and cubic equations.

The solution of such equations is however rarely required in practical work, so that the method is not of much practical importance.

151. *To obtain the numerical values of the roots of a* **quadratic equation.**

I. Let the equation be $x^2 - 2px + q = 0$, where p and q are positive.

Solving, we obtain

$$\begin{aligned} x &= p \pm \sqrt{(p^2 - q)} \\ &= p\left\{1 \pm \sqrt{\left(1 - \frac{q}{p^2}\right)}\right\}. \end{aligned}$$

First, let q be less than p^2; then we can find from the Tables an angle a such that $\sin^2 a = \dfrac{q}{p^2}$.

Whence we obtain

$$x = p\{1 \pm \cos a\}.$$

Secondly, let q be greater than p^2; then we can find from the Tables an angle a such that $\sec^2 a = \frac{q}{p^2}$, then

$$x = p\{1 \pm \sqrt{(-1)} \tan a\}.$$

II. Let the equation be $x^2 + 2px + q = 0$. Then the roots of this equation are equal to those in Case I. with the signs changed.

III. Let the equation be $x^2 - 2px - q = 0$.

Solving, we obtain

$$x = p\left\{1 \pm \sqrt{\left(1 + \frac{q}{p^2}\right)}\right\}.$$

We can find from the Tables an angle a such that

$$\tan^2 a = \frac{q}{p^2},$$

and then $$x = p\{1 \pm \sec a\}.$$

IV. The roots of $x^2 + 2px - q = 0$ are equal to those of Case III. with the sign changed.

Example. Calculate the value of the roots of the equation

$$x^2 - 3{\cdot}4651x - 7{\cdot}6812 = 0.$$

Solving, $$x = \frac{3{\cdot}4651}{2}\left\{1 \pm \sqrt{\left(1 + \frac{4 \times 7{\cdot}6842}{(3{\cdot}4651)^2}\right)}\right\}.$$

Hence, $$\tan^2 a = \frac{4 \times 7{\cdot}6842}{(3{\cdot}4651)^2}; \quad \therefore \tan a = \sqrt{\frac{30{\cdot}7368}{(3{\cdot}4651)^2}};$$

$$L \tan a = \tfrac{1}{2} \log 30{\cdot}7368 - \log 3{\cdot}4651 + 10$$
$$= 10{\cdot}2041135$$
$$= L \tan 57^\circ\, 59'\, 39'',$$
$$x = 1{\cdot}73255\{1 \pm 1{\cdot}60001\}.$$

152. The student will observe that this method is the same as that of adapting the expression $p\left\{1 \pm \sqrt{\left(1 + \frac{q}{p^2}\right)}\right\}$ to Logarithmic calculation by means of the Trigonometrical Tables.

EXAMPLES. XLIX.

Solve the equations:

(i) $x^2+3\cdot416x-8\cdot794=0.$

(ii) $x^2-7\cdot941x+2\cdot7001=0.$

153. *To obtain the numerical value of the roots of a* **cubic equation.**

Let the equation be

$$x^3+3px^2+3qx+r=0.$$

Write $y-p$ for x, and the equation becomes

$$y^3-3(p^2-q)y+(2p^3-3pq+r)=0.$$

Therefore any cubic equation can be transformed into another in which the second term is wanting.

Hence we may take as our standard equation

$$x^3-3ax+b=0$$

To solve this, write ny for x, thus

$$n^3y^3-3any+b=0,$$

or
$$y^3-\frac{3ay}{n^2}+\frac{b}{n^3}=0$$

But, if α be any angle, we have (E. 167)

$$\cos^3\alpha-\tfrac{3}{4}\cos\alpha-\tfrac{1}{4}\cos 3\alpha=0.$$

Hence, if we find α such that

$$\cos 3\alpha=-\frac{4b}{n^3},\quad\text{while}\quad n=2\sqrt{a},$$

then $\cos\alpha$ is one of the roots of the equation.

Also since $\cos(2n\pi\pm 3\alpha)=\cos 3\alpha$, the other two values of y are $\cos(\frac{2}{3}\pi+\alpha)$ and $\cos(\frac{2}{3}\pi-\alpha)$.

But $x=ny$. Therefore the required roots are

$$2(a)^{\frac{1}{2}}\cos\alpha,\quad 2(a)^{\frac{1}{2}}\cos(\tfrac{2}{3}\pi+\alpha),\quad 2(a)^{\frac{1}{2}}\cos(\tfrac{2}{3}\pi-\alpha).$$

3α can be found provided $\frac{1}{2}ba^{-\frac{3}{2}}$ is less than unity, *i.e.* provided b^2 is less than $4a^3$.

EXAMPLES. L.

(1) Solve the following equations:

(i) $x^3 - 3x + 1 = 0$.

(ii) $x^3 - \frac{3}{2}x - \frac{1}{2} = 0$.

(iii) $x^3 - 3x^2 + 3 = 0$.

(iv) $x^3 + 12x^2 + 42x + 44 = 0$.

(v) $x^3 - 3\sqrt{(3)}x^2 - 3x + \sqrt{(3)} = 0$.

(2) Solve the equations.

(i) $x^3 - 439x - 101 = 0$.

(ii) $x^3 - 17651x - 371462 = 0$.

(iii) $1001x^3 - 18472x - 7941 = 0$.

(3) Adapt the following expressions to logarithmic computation:

(i) $a \pm b$. (ii) $a \cos\theta \pm b \sin\theta$.

(iii) $\sin A + \sin B + \sin C - \sin(A + B + C)$.

(iv) $1 + \cos(2\theta - 2m\theta) - \cos(2\theta - 2a) - \cos(2m\theta - 2a)$.

(v) $a \cos A + b \cos B + c \cos C$, where A, B, C are the angles of a triangle.

(4) If $ke^{i\theta}$ be one of the roots of the equation $x^3 + 3qx + r = 0$ prove that $3q = -k^2(1 + 2\cos 2\theta)$ and $r = 2k^3 \cos\theta$.

(5) Prove that in the cubic equation

$$x^3 + 3h^2x + g = 0,$$

if two roots are imaginary, the real root is

$$2h \sinh a \text{ where } \sinh 3a = -\tfrac{1}{2}gh^{-3}.$$

(6) By considering the equation $x^5 + 1 = 0$, prove that,

$$\cos 36^0 = \tfrac{1}{4}(\sqrt{5} + 1).$$

154. We shall conclude this chapter with some examples of Elimination.

Example i. *Eliminate θ from the equations*

$$a\cos\theta+b\sin\theta=c,$$

$$a'\cos\theta+b'\sin\theta=c'.$$

Solving these equations, we obtain

$$\cos\theta=\frac{b'c-bc'}{ab'-a'b},\quad \sin\theta=\frac{c'a-ca'}{ab'-a'b}.$$

But $$\cos^2\theta+\sin^2\theta=1.$$

Therefore $$(b'c-bc')^2+(c'a-ca')^2=(ab'-a'b)^2.$$

This is the required result of elimination.

Example ii. *Prove that the result of eliminating θ from the equations*

$$x\cos\theta-y\sin\theta=a\cos 2\theta,$$

$$x\sin\theta+y\cos\theta=2a\sin 2\theta,$$

is $$(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}.$$

Solving these equations for x and y, we obtain

$$x=a\cos 2\theta\cos\theta+2a\sin 2\theta\sin\theta$$

$$=a\cos^3\theta+3a\sin^2\theta\cos\theta,$$

$$y=2a\sin 2\theta\cos\theta-a\cos 2\theta\sin\theta$$

$$=3a\cos^2\theta\sin\theta+a\sin^3\theta;$$

$$\therefore\ x+y=a(\cos\theta+\sin\theta)^3,$$

$$x-y=a(\cos\theta-\sin\theta)^3;$$

$$\therefore\ (x+y)^{\frac{2}{3}}=a^{\frac{2}{3}}\{1+2\cos\theta\sin\theta\},$$

$$(x-y)^{\frac{2}{3}}=a^{\frac{2}{3}}\{1-2\cos\theta\sin\theta\}.$$

Whence, by addition, we have

$$(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}.$$

EXAMPLES. LI.

(1) Given that $\frac{x^2}{a^2}\cos\theta = \frac{y^2}{a^2}\cos\theta + \frac{z^2}{b^2}\cos\phi$,

and
$$\frac{x}{\sin(\theta+\phi)} = \frac{y}{\sin(\theta-\phi)} = \frac{z}{\sin 2\theta},$$

prove that
$$\frac{\sin\theta}{\sin\phi} = \frac{b^2}{a^2}.$$

(2) Eliminate θ from the equations
$$x = 2a\cos\theta\cos 2\theta - a\cos\theta,$$
$$y = 2b\cos\theta\sin 2\theta - b\sin\theta.$$

(3) Eliminate α and β from the equations
$$x = (a\sin^2\alpha + b\cos^2\alpha)\cos^2\beta + c\sin^2\beta,$$
$$y = a\cos^2\alpha + b\sin^2\alpha, \quad z = (b-a)\sin\alpha\cos\alpha\cos\beta.$$

(4) Eliminate θ from the equations
$$x + a = a(2\cos\theta - \cos 2\theta), \quad y = a(2\sin\theta - \sin 2\theta).$$

(5) Eliminate θ from the equations
$$b\cos^2\theta + a\sin^2\theta = 0, \quad b\cos^2(\theta+\phi) + a\sin^2(\theta+\phi) = 0.$$

(6) Eliminate θ from the equations
$$x = a(\cos\theta + \cos 2\theta),$$
$$y = b(\sin\theta + \sin 2\theta)$$

(7) Eliminate θ from the equations
$$(a+b)\tan(\theta-\phi) = (a-b)\tan(\theta+\phi),$$
$$a\cos 2\phi + b\cos 2\theta = c.$$

(8) Eliminate θ and ϕ from the equations
$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2, \quad \frac{ax}{\cos\phi} - \frac{by}{\sin\phi} = a^2 - b^2,$$
$$\theta - \phi = \tfrac{1}{2}\pi.$$

(9) Eliminate θ from the equations
$$\frac{\cos^3\theta}{\cos(\alpha-3\theta)} = \frac{\sin^3\theta}{\sin(\alpha-3\theta)} = m.$$

(10) Eliminate α from the equations
$$\frac{\sin\theta}{\beta^2-1} = \frac{\cos\theta}{2\beta\sin 2\alpha} = \frac{1}{1+2\beta\cos 2\alpha+\beta^2},$$

shewing that
$$\beta = \tan\tfrac{1}{2}(\pi + 2\theta).$$

MISCELLANEOUS EXAMPLES. LII.

(1) If
$$a\cos(\theta+\phi)+b\cos(\theta-\phi)+c=0,$$
$$a\cos(\phi+\psi)+b\cos(\phi-\psi)+c=0,$$
$$a\cos(\psi+\theta)+b\cos(\psi-\theta)+c=0,$$
then $a^2-b^2+2bc=0$ or $\cos^2\theta=\cos^2\phi=\cos^2\psi=\dfrac{a+b-c}{2a}$.

(2) Prove that when
$$\cos^3(\alpha+\theta)+\cos^3(\beta+\theta)+\cos^3(\gamma+\theta)-3\cos(\alpha+\theta)\cos(\beta+\theta)\cos(\gamma+\theta)$$
vanishes for all values of θ, then either
$$\sin(\beta-\gamma)=\sin(\gamma-\alpha)=\sin(\alpha-\beta)=0,$$
or
$$\cos(\beta-\gamma)=\cos(\gamma-\alpha)=\cos(\alpha-\beta)=-\tfrac{1}{2}$$

(3) Prove that the equations
$$\left(x+\frac{1}{x}\right)\sin\alpha=\frac{y}{z}+\frac{z}{y}+\cos^2\alpha,$$
$$\left(y+\frac{1}{y}\right)\sin\alpha=\frac{z}{x}+\frac{x}{z}+\cos^2\alpha,$$
$$\left(z+\frac{1}{z}\right)\sin\alpha=\frac{x}{y}+\frac{y}{x}+\cos^2\alpha,$$
are not independent; and that they are equivalent to
$$x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=-\sin\alpha,$$

(4) If α and β are the two values of $\sin\theta$ satisfying the equation
$$a\cos 2\theta+b\sin 2\theta=c$$
prove that
$$\alpha^2+\beta^2=\frac{a^2-ac+b^2}{a^2+b^2}.$$

(5) If $\sin(\theta+\alpha)=\sin(\phi+\alpha)=\sin\beta$ and
$$a\sin(\theta+\phi)+b\sin(\theta-\phi)=c,$$
then either $a\sin(2\alpha\pm 2\beta)=-c$ or $a\sin 2\alpha\pm b\sin 2\beta=c$.

(6) Eliminate θ from the equations
$$4(\cos\alpha\cos\theta+\cos\phi)(\cos\alpha\sin\theta+\sin\phi)$$
$$=4(\cos\alpha\cos\theta+\cos\psi)(\cos\alpha\sin\theta+\sin\psi)$$
$$=(\cos\phi-\cos\psi)(\sin\phi-\sin\psi),$$
proving that $\cos(\phi-\psi)=1$ or $\cos 2\alpha$.

(7) If
$$x=2\cos(\beta-\gamma)+\cos(\theta+\alpha)+\cos(\theta-\alpha)$$
$$=2\cos(\gamma-\alpha)+\cos(\theta+\beta)+\cos(\theta-\beta)$$
$$=-2\cos(\alpha-\beta)-\cos(\theta+\gamma)-\cos(\theta-\gamma),$$
prove that $x=\sin^2\theta$, provided that the difference between any two of the angles α, β, γ neither vanishes nor = a multiple of 2π.

(8) If $\dfrac{\sin(\alpha+\theta)}{\sin(\alpha+\phi)} = \dfrac{\sin(\beta+\theta)}{\sin(\beta+\phi)}$ prove that either α and β or θ and ϕ differ by a multiple of π.

(9) If $\dfrac{\sin(\alpha+\theta)}{\sin(\alpha+\phi)} + \dfrac{\sin(\beta+\theta)}{\sin(\beta+\phi)} = \dfrac{\cos(\alpha+\theta)}{\cos(\alpha+\phi)} + \dfrac{\cos(\beta+\theta)}{\cos(\beta+\phi)} = 2,$

prove that either α and β differ by an odd multiple of $\frac{1}{2}\pi$, or θ and ϕ differ by an even multiple of π.

(10) If $(A+B+C)=\pi$ and if $\cos 2A = \dfrac{(d-a)(b-c)}{(d+a)(b+c)}$,

$$\cos 2B = \frac{(d-b)(c-a)}{(d+b)(c+a)}, \quad \cos 2C = \frac{(d-c)(a-b)}{(d+c)(a+b)},$$

then $$\tan A + \tan B + \tan C = \pm 1.$$

(11) If $$\cos\alpha = \cos\beta\cos\phi = \cos\beta'\cos\phi',$$

and $$\sin\alpha = 2\sin\frac{\phi}{2}\sin\frac{\phi'}{2},$$

then $$\tan\frac{\alpha}{2} = \tan\frac{\beta}{2}\,.\,\tan\frac{\beta'}{2}$$

(12) If $\tan\phi = \dfrac{\sin\alpha\sin\theta}{\cos\theta - \cos\alpha}$, then $\tan\theta = \dfrac{\sin\alpha\sin\phi}{\cos\phi \pm \cos\alpha}$.

(13) If β and γ be two values of θ which satisfy

$$\frac{1}{a}\cos\theta + \frac{1}{b}\sin\theta = \frac{1}{c},$$

then $$a\cos\frac{\beta+\gamma}{2} = b\sin\frac{\beta+\gamma}{2} = c\cos\frac{\beta-\gamma}{2}.$$

(14) Given $$a^2\cos\alpha\cos\beta + a(\sin\alpha+\sin\beta) + 1 = 0,$$
$$a^2\cos\alpha\cos\gamma + a(\sin\alpha+\sin\gamma) + 1 = 0,$$

prove that $$a^2\cos\beta\cos\gamma + a(\sin\beta+\sin\gamma) + 1 = 0,$$

and that $$\cos\alpha + \cos\beta + \cos\gamma = \cos(\alpha+\beta+\gamma),$$

β and γ being unequal and less than π.

(15) If θ_1 and θ_2 are two values of θ which satisfy

$$1 + \frac{\cos\theta\cos\phi}{\cos^2\alpha} + \frac{\sin\theta\sin\phi}{\sin^2\alpha} = 0,$$

shew that θ_1 and θ_2 if substituted for θ and ϕ in the equation will satisfy it.

(16) Solve the equations

$$\cos(\theta+\alpha)=\sin\phi\sin\beta,$$
$$\cos(\phi+\beta)=\sin\theta\sin\alpha,$$

and shew that if ϕ_1 and ϕ_2 be the two values of ϕ,

$$\tan(\phi_1+\phi_2)=\frac{\sin 2\beta}{\sin^2\beta-\cos^2\beta\sec^2\alpha}.$$

(17) If $$\cos(\alpha+\theta)+m\cos\theta=n,$$

n^2 cannot be greater than $1+2m\cos\alpha+m^2$.

(18) Eliminate θ and ϕ from the equations

$$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1,\quad \frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1,$$

$$\theta-\phi=2\alpha.$$

proving that $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2}{1+\cos 2\alpha}.$$

GENERAL MISCELLANEOUS EXAMPLES. LIII.

N.B.—For convenience in printing, some writers use $n!$ to denote $1 \,.\, 2 \,.\, 3 \ldots n$.

(1) A person walks from one end A of a wall a certain distance a towards the West, and observes that the other end B then bears E.S.E. He afterwards walks from the end B a distance $(\sqrt{2}+1)\,a$ towards the South, and finds that the end A bears N.W. Shew that the wall makes an angle $\cot^{-1} 2$ with the East.

(2) A man on the top of a hill observes the angles of depression α, β, γ of three consecutive milestones on a straight horizontal road running directly towards him; prove that the height of the hill is

$$1760 \left(\frac{2}{\cot^2 \alpha - 2\cot^2 \beta + \cot^2 \gamma}\right)^{\frac{1}{2}} \text{ yards.}$$

(3) $\sin 2(\beta+\gamma) + \sin 2(\gamma+\alpha) + \sin 2(\alpha+\beta)$

$$= 4 \sin\alpha \sin\beta \sin\gamma \cos(\alpha+\beta+\gamma) + 4\cos\alpha\cos\beta\cos\gamma \sin(\alpha+\beta+\gamma).$$

(4) If $2\alpha + 2\beta + 2\gamma = n\pi$,

then $\sin 2(\beta+\gamma) + \sin 2(\gamma+\alpha) + \sin 2(\alpha+\beta)$

$$= 2(-1)^{\frac{n-1}{2}}\{1-(-1)^n\}\cos\alpha\cos\beta\cos\gamma$$
$$+ 2(-1)^{\frac{n}{2}}\{1+(-1)^n\}\sin\alpha\sin\beta\sin\gamma.$$

(5) $$\frac{\log_a \log_a N}{\sqrt{\log_a b}} - \frac{\log_b \log_b N}{\sqrt{\log_b a}} = \frac{\log_a \log_a b}{\sqrt{\log_a b}} = -\frac{\log_b \log_b a}{\sqrt{\log_b a}}.$$

(6) $$\frac{1}{\underline{|2}} + \frac{4}{\underline{|3}} + \frac{9}{\underline{|4}} + \frac{16}{\underline{|5}} + \ldots = e - 1$$

(7) If $A + B + C = \pi$, and

$$\sin^3\theta = \sin(A-\theta)\sin(B-\theta)\sin(C-\theta),$$

then will $$\cot\theta = \cot A + \cot B + \cot C.$$

(8) Eliminate θ from the equations

$$(a+b)(x+y) = \cos\theta\,(1+2\sin^2\theta),$$
$$(a-b)(x-y) = \sin\theta\,(1+2\cos^2\theta).$$

(9) If $\cos(\theta-\phi)$ is a mean proportional between $\cos\{\pi-(\theta+\phi)\}$ and $\sin(\theta+\phi)$, then

$$\operatorname{cosec} 2\theta+\sec 2\theta=\operatorname{cosec} 2\phi+\sec 2\phi.$$

(10) If $A+B+C=90^0$, then $\operatorname{cosec} A \operatorname{cosec} B \operatorname{cosec} C-2$

$$=\cot B\tan C+\cot C\tan B+\cot C\tan A+\cot A\tan C+\cot A\tan B+\cot B\tan A.$$

(11) Prove that

$$\tan^{-1}\left(\frac{2\cos\alpha\cos\beta\cos\gamma}{\cos^2\alpha+\cos^2\beta+\cos^2\gamma-1}\right)+\tan^{-1}[\tan\sigma\tan(\sigma-\alpha)\tan(\sigma-\beta)\tan(\sigma-\gamma)]=\tan^{-1}1,$$

where $$2\sigma=\alpha+\beta+\gamma.$$

(12) If $4(\alpha+\beta+\gamma)=\pi$, prove that

$$\cos(6\beta+4\gamma-8\alpha)+\cos(6\gamma+4\alpha-8\beta)+\cos(6\alpha+4\beta-8\gamma)$$
$$=4\cos(5\alpha-2\beta-\gamma)\cos(5\beta-2\gamma-\alpha)\cos(5\gamma-2\alpha-\beta).$$

(13) If $$1-x^2-y^2-z^2=2xyz,$$

prove by trigonometry, that

$$x\sqrt{(1-x^2)}+y\sqrt{(1-y^2)}+z\sqrt{(1-z^2)}=2\sqrt{\{(1-x^2)(1-y^2)(1-z^2)\}}.$$

(14) The formulae

$$(2n+\tfrac{1}{4})\pi\pm\alpha,\quad (n-\tfrac{1}{4})\pi+(-1)^n(\tfrac{1}{2}\pi-\alpha)$$

represent the same series of angles.

(15) If $A+B+C=(2m+1)180^\circ$ or $2m180^\circ+90^\circ$, then

$$(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)=2\sin A\,.\,\sin B\,.\,\sin D+2\cos A\cos B\,.\,\cos C+1;$$

and if $A+B+C=2m180^\circ$ or $2m180^\circ-90^\circ$, then

$$(\sin A+\cos A)(\sin B+\cos B)(\sin C+\cos C)=2\sin A\sin B\sin C+2\cos A\cos B\cos C-1.$$

(16) Prove that if

$$b^2+c^2-2bc\cos\alpha=c^2+a^2-2ca\cos\beta=a^2+b^2-2ab\cos\gamma=d^2$$

and $$\alpha+\beta+\gamma=n\pi$$

then $$d^4-d^2(a^2+b^2+c^2)+a^4+b^4+c^4-c^2b^2-c^2a^2-a^2b^2=0.$$

(17) If $\sin 2\theta+\sin 2\phi=\sin 2\alpha$, prove that the three expressions

$$\cos\theta\cos(\alpha+\theta)\sin\theta\sin(\alpha-\theta),$$
$$\cos\phi\cos(\alpha+\phi)\sin\phi\sin(\alpha-\phi),$$
$$\cos\theta\cos\phi\sin\theta\sin\phi,$$

are equal to one another.

(18) Prove that if a, b, c, d be the sides of a quadrilateral, and $2w$ be the sum of two opposite angles, the area of the quadrilateral is

$$\sqrt{(A^2 - abcd\cos^2 w)},$$

where A is the area of a quadrilateral with equal sides but inscribable in a circle.

(19) If $\dfrac{\cos(\alpha+\beta+\theta)}{\sin(\alpha+\beta)\cos^2\gamma} = \dfrac{\cos(\gamma+\alpha+\theta)}{\sin(\gamma+\alpha)\cos^2\beta}$ and β and γ are unequal,

then each of these fractions is equal to $\dfrac{\cos(\gamma+\beta+\theta)}{\sin(\gamma+\beta)\cos^2\alpha}$, and

$$\cot\theta = \frac{\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)}{\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta)+\sin^2(\alpha+\beta+\gamma)}.$$

(20) If
$$\sqrt{2}\cos A = \cos B + \cos^3 B,$$
$$\sqrt{2}\sin A = \sin B - \sin^3 B,$$
then
$$\pm\sin(B-A) = \cos 2B = \tfrac{1}{3}.$$

(21) If $4\cos(x-y)\cos(y-z)\cos(z-x)=1$, prove that

$$1+12\cos 2(x-y)\cos 2(y-z)\cos 2(z-x)$$
$$= 4\cos 3(x-y)\cos 3(y-z)\cos 3(z-x).$$

(22) If

$$\sin(\beta+\gamma) - k\sin(\alpha+\delta) = \sin(\gamma+\alpha) - k\sin(\beta+\delta)$$
$$= \sin(\alpha+\beta) - k\sin(\gamma+\delta),$$

where α, β, γ are unequal and each less than π, then will $k^2=1$, and each member of the equations $=0$.

(23) If $\dfrac{\cos(\beta+\alpha)+\cos(\alpha+\gamma)}{\cos(\beta-\alpha)+\cos(\alpha-\gamma)} = \dfrac{\cos(\beta+\gamma)+\cos(\alpha+\gamma)}{\cos(\gamma-\beta)+\cos(\alpha-\gamma)}$,

then
$$\frac{\sin\beta}{\cos\gamma-\cos\alpha} + \frac{\cos\beta}{\sin\alpha-\sin\gamma} = \frac{1}{\sin\frac{1}{2}(\alpha-\gamma)}.$$

(24) If
$$x^2\cos\alpha\cos\beta + x(\sin\alpha+\sin\beta)+1=0,$$
and
$$x^2\cos\beta\cos\gamma + x(\sin\beta+\sin\gamma)+1=0,$$
prove that
$$x^2\cos\gamma\cos\alpha + x(\sin\gamma+\sin\alpha)+1=0.$$

(25) If
$$x + y\cos\alpha + z\sin\alpha = \cos(\beta-\gamma),$$
$$x + y\cos\beta + z\sin\beta = \cos(\gamma-\alpha),$$
$$x + y\cos\gamma + z\sin\gamma = \cos(\alpha-\beta),$$

prove that $x = 4\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)$.

(26) If
$$a\cos\alpha+b\cos\beta+c\cos\gamma=0,$$
$$a\sin\alpha+b\sin\beta+c\sin\gamma=0,$$
$$a\sec\alpha+b\sec\beta+c\sec\gamma=0,$$
then
$$a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2=0.$$

(27) If
$$a\cos\theta+b\sin\theta+c=0,$$
$$a\cos\phi+b\sin\phi+c=0,$$
$$c\sqrt{2}=\pm a\pm b,$$
prove that either θ or ϕ must be of the form $\frac{1}{2}n\pi+\frac{1}{4}\pi$

(28) If m is a positive integer, then
$$\cos(m+1)\theta=\cos m\theta\left\{2\cos\theta-\frac{1}{2\cos\theta-}\,\frac{1}{2\cos\theta-}\cdots\frac{1}{\cos\theta}\right\}.$$
where $2\cos\theta$ is repeated m times

(29) If $2s=x+y+z$, prove

(i) $\tan(s-x)+\tan(s-y)+\tan(s-z)-\tan s$
$$=\frac{4\sin x\sin y\sin z}{1-\cos^2x-\cos^2y-\cos^2z-2\cos x\cos y\cos z},$$

(ii) $\tan^{-1}(s-x)+\tan^{-1}(s-y)+\tan^{-1}(s-z)-\tan^{-1}s$
$$=\tan^{-1}\frac{16xyz}{(x^2+y^2+z^2+4)^2-4(y^2z^2+z^2x^2+x^2y^2)}.$$

(30) (i)
$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}+\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}=1,$$

(ii)
$$\frac{\sin(\theta-\alpha)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}+\frac{\sin(\theta-\beta)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}+\frac{\sin(\theta-\gamma)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}=0.$$

(31) If a^2, b^2, c^2, are in A. P., then $\tan A$, $\tan B$, $\tan C$ are in H. P.

(32) If $\dfrac{\tan A}{x}=\dfrac{\tan B}{y}=\dfrac{\tan C}{z}$ and $A+B+C=180°$, prove that each fraction is equal to
$$\sin A\,.\,\sin B\,.\,C\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-\frac{1}{x+y+z}\right).$$

(33) $OI^2+OI_1^2+OI_2^2+OI_3^2=12R^2$.

(34) If α, β, γ are the radii of the circles OBC, OCA, OAB, then

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=\frac{abc}{R^3}.$$

(35) The sides of a triangle are

$$x^2+xy+y^2,\quad 2xy+x^2,\quad x^2-y^2;$$

prove that the angles are in A.P., the common difference being

$$2\tan^{-1}\left(\frac{y\sqrt{3}}{2x+y}\right).$$

(36) $4R(r_2r_3+r_3r_1+r_1r_2)=(r_2+r_3)(r_3+r_1)(r_1+r_2)$

(37) $\Delta=s^2 2^{\frac{3}{2}}\cos\theta\cos\phi\cos\psi$, where

$$\cos 2\theta=\tan\tfrac{1}{2}B+\tan\tfrac{1}{2}C,$$
$$\cos 2\phi=\tan\tfrac{1}{2}C+\tan\tfrac{1}{2}A,$$
$$\cos 2\psi=\tan\tfrac{1}{2}A+\tan\tfrac{1}{2}B.$$

(38) $4(AA'^2+BB'^2+CC'^2)=3(a^2+b^2+c^2)$.

$16(BB'^2 . CC'^2+CC'^2 . AA'^2+AA'^2 . BB'^2)=9(b^2c^2+c^2a^2+a^2b^2)$.

$16(AA'^4+BB'^4+CC'^4)=9(a^4+b^4+c^4)$. [Art. 129.]

(39) If a, b, c, d be the lengths of the sides of a quadrilateral such that one circle can be described about it and another inscribed in it, then the radius of the latter circle is

$$\frac{2\sqrt{(abcd)}}{a+b+c+d}.$$

(40) Given that $x=y\cos Z+z\cos Y$,

$$y=z\cos X+x\cos Z,$$

and that $X+Y+Z$ is an odd multiple of 180°, then

$$z=x\cos Y+y\cos X,$$

and $$\cos X=\frac{y^2+z^2-x^2}{2yz}.$$

(41) If $A+B+C=180°$ and

$$\frac{y\sin C-z\sin B}{x-y\cos C-z\cos B}=\frac{z\sin A-x\sin C}{y-z\cos A-x\cos C},$$

x, y, z being real, then $\dfrac{x}{\sin A}=\dfrac{y}{\sin B}=\dfrac{z}{\sin C}$.

(42) In any triangle

$$R^2(a^4+b^4+c^4-2b^2c^2\cos 2A-2c^2a^2\cos 2B-2a^2b^2\cos 2C)=2a^2b^2c^2.$$

(43) The radii of the escribed circles of a triangle are the roots of the equation

$$Sx^3+s(\Sigma-s^2)x^2+s^2Sx=S^2s,$$

where 2Σ is the sum of the squares of the sides.

(44) $12RS=a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)$.

(45) If O be a point within a triangle, such that AO, BO, CO are inversely proportional to the sides BC, CA, AB; R, R_1, R_2, R_3 are the radii of the circles described about ABC, BOC, COA, AOB; then

$$\left(\frac{2}{R_1}-\frac{1}{R}\right)\tan A=\left(\frac{2}{R_2}-\frac{1}{R}\right)\tan B=\left(\frac{2}{R_3}-\frac{1}{R}\right)\tan C.$$

(46) ABC is a triangle in which the sides AB, AC are equal. Circles are described with centres A, B, C touching each other externally. Prove that the distance between the centres of the circles that can be drawn touching these three circles is

$$4b\,\frac{(1-\cos B)(1-2\cos B)}{4-5\cos B}.$$

(47) Perpendiculars OD, OE, OF to the sides of a triangle when produced backwards meet the circumscribing circle in P, Q, R; prove that $\quad 4r(PD+QE+RF)=2bc+2ca+2ab-a^2-b^2-c^2.$

(48) If ρ_1, ρ_2, ρ_3 are the distances of any point in the plane of an equilateral triangle, whose side is a, from the angular points, then

$$\rho_2^2\rho_3^2+\rho_3^2\rho_1^2+\rho_1^2\rho_2^2-\rho_1^4-\rho_2^4-\rho_3^4+a^2(\rho_1^2+\rho_2^2+\rho_3^2)=a^4.$$

(49) From the angular points of a triangle ABC are drawn perpendiculars to the opposite sides and also lines bisecting the angles: if θ be the angle between the two lines drawn from A, and ϕ, ψ be corresponding angles at B, C, prove that

$$1+\cos\theta+\cos\phi+\cos\psi=4\cos\tfrac{1}{2}(B-C)\cos\tfrac{1}{2}(C-A)\cos\tfrac{1}{2}(A-B).$$

(50) On the sides of a scalene triangle ABC as bases similar isosceles triangles are described either all internally or all externally, and their vertices joined so as to form a new triangle $A'B'C'$, then if $A'B'C'$ is equilateral the angles at the bases of the isosceles triangles will be each 30^0, and if it is similar to ABC they are each

$$\tan^{-1}\frac{4\Delta}{a^2+b^2+c^2}.$$

(51) Three circles touching each other externally are all touched by a fourth circle including them all. If a, b, c are the radii of the internal circles and α, β, γ the distance of their centres from that of the external circle, then

$$2\left(\frac{\beta\gamma}{bc}+\frac{\gamma\alpha}{ca}+\frac{\alpha\beta}{ab}\right)=4+\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}.$$

(52) Circles are described on the sides a, b, c of a triangle as diameters; prove that the diameter D of the circle which touches them externally is such that

$$\left(\frac{D}{s-a}-1\right)^{\frac{1}{2}}+\left(\frac{D}{s-b}-1\right)^{\frac{1}{2}}+\left(\frac{D}{s-c}-1\right)^{\frac{1}{2}}=\left(\frac{s}{D-s}\right)^{\frac{1}{2}}.$$

(53) If O be any point and ρ_1, ρ_2, ρ_3 the reciprocals of the radii of the circles circumscribing the three triangles OBC, OCA, OAB, prove that

$$(a\rho_1+b\rho_2+c\rho_3)(-a\rho_1+b\rho_2+c\rho_3)(a\rho_1-b\rho_2+c\rho_3)(a\rho_1+b\rho_2-c\rho_3)$$
$$=a^2\rho_1{}^2b^2\rho_2{}^2c^2\rho_3{}^2.$$

(54) Lines drawn parallel to the sides of a triangle ABC through the centres of the circles escribed to that triangle form a triangle $A'B'C'$, prove that the perimeter of the triangle $A'B'C'$ is

$$4R\cot\tfrac{1}{2}A\cot\tfrac{1}{2}B\cot\tfrac{1}{2}C.$$

(55) If BE, CF be the perpendiculars from B and C on to the opposite sides, and if FE and BC produced meet in Q, prove that

$$2(QE^2-QF^2)=(BQ^2-CQ^2)(\cos 2B+\cos 2C).$$

(56) If O be any point within a triangle ABC the sides of which are abc, and if R_1, R_2 be the radii of the circles circumscribing the triangles BOC, COA, prove that

$$\frac{1}{R_1a\alpha}-\frac{1}{R_2b\beta}=\frac{1}{R_3c\gamma}\left(\frac{\beta^2+\gamma^2}{a^2}-\frac{\gamma^2+\alpha^2}{b^2}\right),$$

where $$OA=\alpha,\quad OB=\beta,\quad OC=\gamma.$$

(57) A chord is drawn cutting two concentric circles whose radii are as $1:n$ so that the intercepted portions subtend angles 2α, 2β at the centre; prove that the chord is divided at either point of intersection with the inner circle in the ratio

$$n^2-2n\cos(\alpha+\beta)+1:n^2-1.$$

(58) A straight line cuts three concentric circles in A, B, C and passes at a distance p from their centre. The area of the triangle formed by the tangents at ABC is $\dfrac{BC\,.\,CA\,.\,AB}{2p}$.

(59) A polygon of $3n$ sides, which are a, b, c successively, repeated n times, is inscribed in a circle. If the angular points be A, B, C, D, E, etc., and the radius of the circle is denoted by r, prove that

$$AC^2 = \left\{ac + 2br \sin\frac{\pi}{n}\right\}\left\{bc + 2ar \sin\frac{\pi}{n}\right\} \div \left\{ab + 2cr \sin\frac{\pi}{n}\right\}.$$

(60) The tangent at the point of contact of the inscribed circle and the circle of nine points of the triangle ABC cuts the side AC at a distance from A

$$\frac{b\,(a-b)}{a-2b+c}.$$

(61) $\tan x = kx$ has an infinity of roots.

(62) The equation $\theta = \cos\theta$ has one and only one solution, such that the value of θ is less than $\frac{1}{2}\pi$.

(63) $8 \sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C$ is less than 1 except when the triangle ABC is equilateral.

(64) If $A + B + C = 90^0$, the least value of

$$\tan^2 A + \tan^2 B + \tan^2 C \text{ is } 1.$$

(65) If $A + B + C = 90^0$

$$\tan^2 B \tan^2 C + \tan^2 C \tan^2 A + \tan^2 A \tan^2 B$$

is always less than 1; and if one angle approach indefinitely near to two right angles the least value of the expression is $\frac{1}{2}$.

(66) If $A + B + C = 180^0$, the least value of

$$\cot^2 A + \cot^2 B + \cot^2 C \text{ is } 1.$$

(67) In any triangle $\cos A + \cos B + \cos C$ is > 1 and not greater than $\frac{3}{2}$.

(68) $\sin A + \sin B + \sin C$ is never less than

$$\sin 2A + \sin 2B + \sin 2C, \text{ if } A + B + C = 180^0.$$

(69) $$\left(2\cos\alpha - \frac{1}{2\cos\alpha -}\,\frac{1}{2\cos\alpha -}\cdots\right)\left(2\cos\beta - \frac{1}{2\cos\beta -}\,\frac{1}{2\cos\beta -}\cdots\right)$$
$$= 2\cos(\alpha+\beta) - \frac{1}{2\cos(\alpha+\beta) -}\,\frac{1}{2\cos(\alpha+\beta) -}$$

(70) Prove by induction that

$$\sin(\alpha+\beta+\gamma+\ldots n \text{ angles}) = (S_1C_{n-1}) - (S_3C_{n-3}) + (S_5C_{n-5}) - \text{etc.}$$
$$\cos(\alpha+\beta+\gamma+\ \ n \text{ angles}) = (C_n) - (S_2C_{n-2}) + (S_4C_{n-4}) - \text{etc.}$$

where (S_rC_{n-r}) stands for the sum of the products of the sines $\sin\alpha$, $\sin\beta$, $\sin\gamma$. taken r together, each multiplied by the product of the $(n-r)$ remaining cosines.

Shew that De Moivre's Theorem is equivalent to these two theorems.

(71) Prove that if n is an odd integer the two series

$$1 - \frac{n(n-1)}{1.2} + \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} - \quad ,$$
$$n - \frac{n(n-1)(n-2)}{1.2.3} + \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5} - \ldots ,$$

are numerically equal, and if n is an even integer one of the two series is zero.

(72) Prove that $\dfrac{3}{1+x^3} = \dfrac{1}{1+x} + \dfrac{1}{1-\alpha x} + \dfrac{1}{1-\beta x}$,

where -1, α, β are the values of $(-1)^{\frac{1}{3}}$, and deduce by writing $x = \cos 2\theta + i\sin 2\theta$ that

$$3\tan 3\theta = \tan\theta - \cot(\theta + \tfrac{1}{6}\pi) - \cot(\theta - \tfrac{1}{6}\pi).$$

(73) If $$\sin\log(a+ib) = \alpha + i\beta,$$

then $$\log\sin(a'+ib') = \alpha' + i\beta';$$

when $$a' = \log\sqrt{(a^2+b^2)},\quad b' = \tan^{-1}\frac{b}{a},$$

$$\alpha' = \log\sqrt{(\alpha^2+\beta^2)},\quad \beta' = \tan^{-1}\frac{\beta}{\alpha}.$$

(74) By writing in the identity

$$\frac{1}{(x-a)(x-b)} = \frac{1}{(a-b)(x-a)} - \frac{1}{(a-b)(x-b)},$$

$\cos 2\theta + \sqrt{(-1)} \sin 2\theta$ for x and similar quantities in terms of α and β for a and b, prove that

$$\cos(2\theta+\alpha+\beta)\sin(\alpha-\beta) = \cos(2\alpha+\theta+\beta)\sin(\theta-\beta) - \cos(2\beta+\theta+\alpha)\sin(\theta-\alpha).$$

(75) Prove that

$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y,$$
$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y.$$

(76) If $2x = 4\cos\alpha\cosh b$, $2y = 4\sin\alpha\sinh b$, then

$$\sec(\alpha+ib) + \sec(\alpha-ib) = \frac{4x}{x^2+y^2},$$

$$\sec(\alpha+ib) - \sec(\alpha-ib) = \frac{4iy}{x^2+y^2}.$$

(77) One of the values of $\sin^{-1}(\cos\theta + i\sin\theta)$ is

$$\cos^{-1}(\sqrt{\sin\theta}) + i\sin^{-1}(\sqrt{\sin\theta})$$

when θ is between 0 and $\frac{1}{2}\pi$.

(78) Reduce $\tan^{-1}(\cos\theta + i\sin\theta)$ to the form $A+iB$, and hence prove that $\cos\theta - \frac{1}{3}\cos 3\theta + \frac{1}{5}\cos 5\theta - \quad = \pm\frac{1}{4}\pi$; the upper or lower sign being taken according as θ is positive or negative.

(79) If ω is an imaginary cube root of -1,

$$\left\{\cos\alpha + \omega\cos\left(\alpha+\frac{\pi}{3}\right) + \omega^2\cos\left(\alpha+\frac{2\pi}{3}\right)\right\} \left\{\cos\beta + \omega\cos\left(\beta+\frac{\pi}{3}\right) + \omega^2\cos\left(\beta+\frac{2\pi}{3}\right)\right\}$$
$$= \frac{3}{2}\left\{\cos(\alpha+\beta) + \omega\cos\left(\alpha+\beta+\frac{\pi}{3}\right) + \omega^2\cos\left(\alpha+\beta+\frac{2\pi}{3}\right)\right\},$$

and deduce the value of

$$\left\{\cos\alpha + \omega\cos\left(\alpha+\frac{\pi}{3}\right) + \omega^2\cos\left(\alpha+\frac{2\pi}{3}\right)\right\}^n,$$

when n is a positive integer.

(80) Prove that the real part of one of the values of $(1+i\tan\theta)^{-i}$ is $e^{\theta}\cos(\log\cos\theta)$.

(81) $(a+\iota b)^{a+\iota\beta}$ will be wholly real or wholly imaginary according as $\frac{1}{2}\beta\log(a^2+b^2)+a\tan^{-1}\frac{b}{a}$ is an even or odd multiple of $\frac{1}{2}\pi$.

(82) Prove that all solutions of the equation $\sinh x=\sinh a$ are included in the expression $x=\iota n\pi+(-1)^n a$ where n is any integer, positive or negative.

(83) Prove the following rule for finding the length of a small circular arc; to 256 times the chord of one-fourth the arc add the chord of the arc; subtract 40 times the chord of half the arc and divide the remainder by 45.

(84) If two sides a, b and the included angle C of a triangle are given, and a small error δ exists in C, the corresponding error in R is

$$\tfrac{1}{2}\delta a\cot A\cos B\operatorname{cosec} C.$$

(85) If the unit of measurement be a right angle, find the limit of $\dfrac{\tan\theta-\sin\theta}{\theta^3}$ as θ is indef. diminished.

(86) The limit when n is indefinitely increased of

$$\left(\cos\frac{a}{n}+\sin\frac{3a}{n}\right)^n \text{ is } e^{3a}.$$

(87) If $\dfrac{\sin\theta}{\theta}=\dfrac{863}{864}$, then $\theta=5^0$ nearly.

(88) If $\theta=\frac{1}{2}\pi$ nearly and n is >1, prove that

$$(\sin\theta)^{\frac{1}{n}}=\frac{n-1+(n+1)\sin\theta}{n+1+(n-1)\sin\theta}\text{ nearly.}$$

(89) If $\cos^{-1}\dfrac{a}{a+b+x}-\cos^{-1}\dfrac{a}{a+x}=\gamma$, and b and x are both small compared with a, then

$$x=\frac{a\sin^2\gamma}{8}\left(1-\frac{2b}{a\sin^2\gamma}\right)^2\text{ nearly.}$$

(90) If in the equation

$$\tan\theta=\frac{1}{\cot\alpha_1+\cot\alpha_2}+\frac{1}{\cot\alpha_3+\cot\alpha_4}$$

the angles α_1, α_2, α_3, α_4 are all nearly equal, then one value of θ is very nearly $\frac{1}{4}(\alpha_1+\alpha_2+\alpha_3+\alpha_4)$.

(91) θ differs from $\tan\theta$ by less than $\frac{1}{3}\tan^3\theta$, θ being less than $\frac{\pi}{4}$.

(92) $$\frac{\tan^2\theta-\sinh^2\alpha}{\tan^2\theta-\sinh^2\beta}=\frac{\coth^2\alpha\sin^2\theta-1}{\coth^2\beta\sin^2\theta-1}\times\frac{\sinh^2\alpha}{\sinh^2\beta}.$$

(93) $$\sin(\alpha-2n\beta)+\sin\{\alpha-2(n-1)\beta\}+\sin\{\alpha-(2n-2)\beta\}+\ldots$$
$$+\sin\{\alpha+2n\beta\}=\sin\alpha\operatorname{cosec}\beta\sin(2n+1)\beta.$$

(94) $$\tan\alpha\tan(\alpha+\beta)+\tan(\alpha+\beta)\tan(\alpha+2\beta)+$$
$$+\tan\{\alpha+(n-1)\beta\}\tan(\alpha+n\beta)=\frac{\tan(\alpha+n\beta)-\tan\alpha-n\tan\beta}{\tan\beta}.$$

(95) $$\sec\alpha\sec(\alpha+\beta)+\sec(\alpha+\beta)\sec(\alpha+2\beta)$$
$$+\sec(\alpha+2\beta)\sec(\alpha+3\beta)$$
$$+\ \ldots \text{ to } n \text{ terms}$$
$$=\operatorname{cosec}\beta\{\tan(\alpha+n\beta)-\tan\alpha\}.$$

(96) Sum the series to n terms and to infinity
$$\tan^{-1}\frac{4}{1+3\cdot4}+\tan^{-1}\frac{6}{1+8\cdot9}+\tan^{-1}\frac{8}{1+15\cdot16}.$$

(97) $$\tan^{-1}\tfrac{1}{3}+\tan^{-1}\tfrac{1}{7}+\tan^{-1}\tfrac{1}{13}+\tan^{-1}\tfrac{1}{21}+\ \ldots \text{ to } n \text{ terms}$$
$$=\tan^{-1}\frac{n}{n+2}.$$

(98) Sum to n terms the series
$$1\cdot\tan\theta\cdot\sec2\theta+\sec\theta\cdot\tan2\theta\sec3\theta+\sec2\theta\tan3\theta\cdot\sec4\theta+$$

(99) $$(1+\sec2\theta)(1+\sec4\theta)(1+\sec8\theta)\ \ldots(1+\sec2^n\theta)$$
$$=\frac{\tan2^n\theta}{\tan\theta}$$

(100) $$(2\cos\theta-1)(2\cos2\theta-1).\ (2\cos2^{n-1}\theta-1)=\frac{2\cos2^n\theta+1}{2\cos\theta+1}$$

(101) Sum the series
$$\frac{\sin\theta}{2\cos\theta+1}+\frac{3\sin3\theta}{2\cos3\theta+1}+\frac{3^2\sin3^2\theta}{2\cos3^2\theta+1}\ \ldots \text{ to } n \text{ terms}$$

(102) The n^{th} convergent of
$$\frac{1}{2\tan\alpha+}\ \frac{1}{2\tan\alpha+}\ \frac{1}{2\tan\alpha+}\ \text{etc}, \text{ is}$$
$$\frac{(\tan\alpha+\sec\alpha)^n-(\tan\alpha-\sec\alpha)^n}{(\tan\alpha+\sec\alpha)^{n+1}-(\tan\alpha-\sec\alpha)^{n+1}}.$$

(103) The sum of n terms of the series

$$1+\frac{\cos\alpha}{\cos\alpha}+\frac{\cos 2\alpha}{\cos^2\alpha}+\ldots=0,\text{ if } n\alpha=\pi.$$

(104) Sum the series

$$\log(1-2\cos\theta)+\log(1-2\cos 2\theta)+\log(1-2\cos 2^2\theta)+\text{etc.}$$

to n terms.

(105) $$\frac{3\sin x-\sin 3x}{\cos 3x}+\frac{3\sin 3x-\sin 3^2x}{3\cos 3^2x}+\frac{3\sin 3^2x-\sin 3^3x}{3^2\cos 3^3x}+\ldots$$

to n terms $=\frac{3}{2}\left(\frac{\tan 3^n x}{3^n}-\tan x\right)$.

(106) The roots of the equation

$$4x^3+8x^2-19x+8=0$$

are $$\sin^2\tfrac{1}{7}\pi,\ \sin^2\tfrac{3}{7}\pi,\ \sin^2\tfrac{5}{7}\pi.$$

(107) Solve the equation $x^3+x^2-2x-1=0$.

[Result. $2\cos\frac{2}{7}\pi$, $2\cos\frac{4}{7}\pi$, $2\cos\frac{6}{7}\pi$.]

(108) $$(x-2\cos\tfrac{2}{5}\pi)(x-2\cos\tfrac{4}{5}\pi)(x-2\cos\tfrac{6}{5}\pi)(x-2\cos\tfrac{8}{5}\pi)$$
$$=x^4+2x^3-x^2-2x+1.$$

(109) The sum to n terms of the series

$$1^2\cos 2\alpha+2^2\cos 4\alpha+3^2\cos 6\alpha+\quad\text{to } n \text{ terms}$$
$$=\frac{n^2}{2}\frac{\sin(2n+1)\alpha}{\sin\alpha}+\frac{n}{2}\frac{\cos 2n\alpha}{\sin^2\alpha}-\frac{1}{4}\frac{\sin 2n\alpha\,.\cos\alpha}{\sin^3\alpha}.$$

(110) Sum the series

$$1+\frac{\theta}{1}\frac{\sin 2\theta}{\sin^2\theta}+\frac{\theta^2}{1\,.\,2}\frac{\sin 3\theta}{\sin^3\theta}+\frac{\theta^3}{1\,.\,2\,.\,3}\frac{\sin 4\theta}{\sin^4\theta}+\quad\text{ad inf.}$$

(111) Sum to n terms

(i) $1+e\sin\alpha+e^2\sin 2\alpha+\text{etc.}$

(ii) $\tan\frac{\theta}{2}\sec\theta+\tan\frac{\theta}{2^2}\sec\frac{\theta}{2}+\text{etc.}$

(112) Sum to infinity

$$n\sin\alpha+\frac{n(n+1)}{1\,.\,2}\sin 2\alpha+\frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}\sin 3\alpha+\ldots$$

(113) Prove that

$$\frac{1-h^2}{(1-h)^2\cos^2\frac{1}{2}x+(1+h)^2\sin^2\frac{1}{2}x}=1+2h\cos x+2h^2\cos 2x+\text{etc.}$$

(114) $$\log\frac{a^2}{a^2\cos^2\theta+b^2\sin^2\theta}=4\left\{\frac{a-b}{a+b}\sin^2\theta-\frac{1}{2}\left(\frac{a-b}{a+b}\right)^2\sin^2 2\theta\right.$$
$$\left.+\frac{1}{3}\left(\frac{a-b}{a+b}\right)^3\sin^2 3\theta-\text{etc.}\right\}$$

(115) $$1+\frac{x^3}{3\,!}+\frac{x^6}{6\,!}+\quad=\frac{1}{3}\{e^x+2e^{-\frac{1}{2}x}\cos\tfrac{1}{2}(x\sqrt{3})\}.$$

(116) The roots of the equation

$$x^n\sin n\alpha-nx^{n-1}\sin(n\alpha+\beta)+\frac{n(n-1)}{2\,!}x^{n-2}\sin(n\alpha+2\beta)-\quad.=0$$

are given by $x=\sin(\alpha+\beta-k\phi)\operatorname{cosec}(\alpha-k\phi)$ where k has all integral values from 0 to $n-1$ and $n\phi=\pi$.

(117) Find the general value of θ which satisfies the equation

$$(\cos\theta+i\sin\theta)(\cos 2\theta+i\sin 2\theta)\quad(\cos n\theta+i\sin n\theta)=1.$$

(118) When n is even and if $n\phi=\pi$,

$$\tan\alpha\tan(\alpha+\phi)\tan(\alpha+2\phi)\quad\tan\{\alpha+(n-1)\phi\}=(-1)^{\frac{n}{2}}.$$

(119) When m is odd

$$\tan m\phi=\tan\phi\cot\left(\phi+\frac{\pi}{2m}\right)\tan\left(\phi+\frac{2\pi}{2m}\right)\,..$$
$$\cot\left(\phi+\frac{m-2}{2m}\pi\right)\tan\left(\phi+\frac{m-1}{2m}\pi\right).$$

(120) $$\tan\theta+\tan\left(\frac{2\pi}{m}+\theta\right)+\tan\left(\frac{4\pi}{m}+\theta\right)+\ \dots$$
$$+\tan\left(2\frac{m-1}{m}\pi+\theta\right)=\frac{m\sin m\theta}{\cos m\theta-S_m(-1)^{\frac{m}{2}}},$$

where $S_m=\frac{1}{2}$ the sum of the m^{th} powers of the root of equation $z^2-1=0$.

(121) If $\theta=\dfrac{\pi}{2^n+1}$,

$$2^n\cos\theta\cos 2\theta\cos 2^2\theta\quad\cos 2^{n-1}\theta=1.$$

(122) If $a, b, c, \ldots$ are the roots of the equation

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - p_3 x^{n-3} + \text{etc.},$$

then $$\tan^{-1} a + \tan^{-1} b + \tan^{-1} c + \ldots = \tan^{-1} \frac{p_1 - p_3 + p_5 - \ldots}{1 - p_2 + p_4 - \ldots}.$$

(123) Prove that

$$e^{\pi} + e^{-\pi} = 2(1 + 2^2)\{1 + (\tfrac{2}{3})^2\}\{1 + (\tfrac{2}{5})^2\}$$

(124) The sum of the products of each pair of the reciprocals of the fourth powers of every positive integer is $\frac{384\pi^8}{5!\,9!}$.

(125) $$\tan\frac{y}{2} = \frac{2}{\pi - y} - \frac{2}{\pi + y} + \frac{2}{3\pi - y} - \frac{2}{3\pi + y} + \frac{2}{5\pi - y} - \frac{2}{5\pi + y}.$$

(126) $$\cot\frac{y}{2} = \frac{2}{y} - \frac{2}{2\pi - y} + \frac{2}{2\pi + y} - \frac{2}{4\pi - y} + \frac{2}{4\pi + y}.$$

(127) Prove that the coefficients of θ^2 and θ^4 in the expression

$$\left\{\theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\ldots\right\}^2 + \left\{\left(1 - \frac{4\theta^2}{\pi^2}\right)\left(1 - \frac{4\theta^2}{3^2\pi^2}\right)\ldots\right\}^2$$

vanish; explaining *à priori* why they do so.

(128) Having given the formula

$$\sin\theta = \theta\left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right) \times \ldots$$

deduce the expression for $\cos\theta$ in factors

(129) The coefficient of x^n in the expansion of

$$(1 + x)\left(1 + \frac{x}{2^2}\right)\left(1 + \frac{x}{3^2}\right)\ldots \text{ is } \frac{\pi^{2n}}{(2n+1)!}.$$

(130) By putting $a \pm ia$ for θ in the expression of $\sin\theta$ in factors, prove that

$$\tan^{-1}\frac{2a^2}{\pi^2} + \tan^{-1}\frac{2a^2}{2^2\pi^2} + \tan^{-1}\frac{2a^2}{3^2\pi^2} + \ldots \text{ad inf.}$$
$$= \tfrac{1}{4}\pi - \tan^{-1}\{\tanh a \cot a\} + n\pi.$$

(131) If a series of points are distributed symmetrically round the circumference of a circle, the sum of the squares of their distances from a point on the circumference is twice that from the centre.

(132) If $A_1, A_2, A_3, \dots A_{2n+1}$ are angular points of a regular polygon inscribed in a circle and O any point in the circumference between A_1 and A_{2n+1}, then the sum of the lengths

$$OA_1 + OA_3 + OA_5 + \dots + OA_{2n+1} = \text{the sum } OA_2 + OA_4 + OA_6 + \quad + OA_{2n}.$$

(133) If from a point P straight lines $PB_1, PB_2, \dots PB_n$ be drawn to the middle points of the sides of a closed polygon $A_1A_2 \dots A_n$, and if the angles PB_1A_1, $PB_2A_2, \dots PB_nA_n$ be denoted by $\alpha_1, \alpha_2, \ . \alpha_n$ respectively, and the triangles PA_1A_2, $PA_2A_3, \dots PA_nA_1$ by $\Delta_1, \Delta_2, \dots \Delta_n$, prove that

$$\Delta_1 \cot \alpha_1 + \Delta_2 \cot \alpha_2 + \dots + \Delta_n \cot \alpha_n = 0.$$

EXAMINATION PAPERS.

I. Sandhurst—Further. *Nov.* 1882.

1. Name and define the trigonometrical ratios. Prove that

$$\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \operatorname{cosec}^2 A.$$

If the cosecant of an angle between 90^0 and 180^0 is $\frac{2}{\sqrt{3}}$, what is the secant? And if the cosine of an angle between 540^0 and 639^0 is $-\frac{1}{2}$, what is the cosecant?

2. Prove the following identities:—

i. $(\sin 2A)^2 = 2\cos^2 A\,(1 - \cos 2A)$.

ii. $2\operatorname{cosec} 4A + 2\cot 4A = \cot A - \tan A$.

iii. $2\tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{5} = \frac{\pi}{4}$.

3. In a plane triangle ABC prove that—

i. $\tan A \tan B \tan C = \tan A + \tan B + \tan C$.

ii. $a\sin A + b\sin B + c\sin C = 2(\alpha\cos A + \beta\cos B + \gamma\cos C)$,

where abc are the sides and $\alpha\beta\gamma$ the perpendiculars let fall on them from the opposite angles respectively.

4. Prove that the area of a triangle

$$= \tfrac{1}{4}a^2\sin 2B + \tfrac{1}{4}b^2\sin 2A;$$

and if R, r are the radii of the circumscribing and inscribed circles

$$Rr = \frac{abc}{4(a+b+c)}.$$

5. Given $\log 1\frac{1}{5} = \cdot0791812$ and $\log 2\frac{2}{5} = \cdot3802112$, find the value of $\sqrt[5]{(3\cdot6)^3} \times \sqrt[4]{\frac{1}{25}} \div \sqrt[3]{8\frac{49}{72}}$, the mantissae for 46929 and 46930 being 6714413 and 6714506.

In a triangle ABC, $b = 14$, $c = 11$, $A = 60^0$; find the other angles, having given $L\tan 11^0\,44'\,29'' = 9\cdot31774$.

6. A measured line is drawn from a point on a horizontal plane in a direction at right angles to the line joining that point to the base of a tower standing on the plane. The angles of elevation of the tower from the two ends of the measured line are 30^0 and 18^0. Find the height of the tower in terms of l, the length of the measured line.

II. WOOLWICH—PRELIMINARY. *June*, 1882.

1. Prove that the angle subtended at the centre of a circle by an arc equal in length to its radius is an invariable angle.

One angle of a triangle is 45^0, and the circular measure of another is $1\frac{1}{2}$. Find the third, both in degrees and in circular measure.

2. Define the secant of an angle, and shew how your definition applies to angles between 180^0 and 270^0.

If $\sec A = -2$, what two values between 0^0 and 360^0 may A have?

3. Obtain a formula embracing all the angles which have a given tangent.

Determine all the values of θ which satisfy the equation;

$$\sqrt{3}\tan^2\theta + 1 = (1+\sqrt{3})\tan\theta.$$

4. Find an expression for $\tan 3A$ in terms of $\tan A$. Shew also that $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$.

5. Prove that $$\sin 18^0 = \frac{\sqrt{5}-1}{4},$$
and that $$\sin^2 30^0 = \sin 18^0 \sin 54^0.$$

Shew that in any circle the chord of an arc of 108^0 is equal to the sum of the chords of arcs of 36^0 and 60^0.

6. Demonstrate the identities—

(1) $$\frac{(\operatorname{cosec} A + \sec A)^2}{\operatorname{cosec}^2 A + \sec^2 A} = 1 + \sin 2A.$$

(2) $\sin 3A = 4\sin A \sin(60^0 + A)\sin(60^0 - A)$.

(3) $4(\cot^{-1} 3 + \operatorname{cosec}^{-1}\sqrt{5}) = \pi$.

7. What are the advantages gained by the use of logarithms calculated to the base 10?

If $\log_{10} 2 = \cdot 30103$, find the logarithms of 5, $\frac{1}{16}$, and $4\sqrt{\cdot 005}$, to the base 10.

8. Prove that in any triangle—

(1) $2bc\cos A = b^2 + c^2 - a^2$.

(2) $$\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B} = \frac{a^2+b^2}{a^2+c^2}.$$

If r_1 be the radius of a circle touching the side a of a triangle and the other two sides produced, shew that—

$$r_1 \cos \frac{A}{2} = a \cos \frac{B}{2} \cos \frac{C}{2}.$$

If a be the side of a regular polygon of n sides, and R, r, the radii respectively of its circumscribed and inscribed circles, prove that

$$R + r = \tfrac{1}{2} a \cot \frac{\pi}{2n}.$$

10. Two sides of a triangle, which are respectively 250 and 200 yards long, contain an angle of $54^\circ 36' 24''$.

Find the two other angles, having given

$L \cot 27^\circ 18' = 10{\cdot}2872338$, diff. for $1' = {\cdot}3100$;

$L \tan 12^\circ 8' 50'' = 9{\cdot}3329292$; $\log 3 = {\cdot}4771213$.

11. The eye of a soldier in a straight trench of uniform depth is 2 feet above a level plain on which he sees two men standing in the same straight line as the trench; the parts of their bodies above the level of his eye subtending at it the angles $\tan^{-1} {\cdot}0041\dot{6}$ and $\tan^{-1} {\cdot}00\dot{4}$. On walking 200 ft. towards them in the trench he notices that the height of one exactly hides that of the other; and, on approaching 596 feet 8 in. closer still he finds that the portion of the height of the nearer above the level of his eye subtends at it 45°. Find the heights of the men.

III. Woolwich—Preliminary. *Dec.* 1882.

1. Shew how to express in degrees, minutes, and seconds, an angle whose circular measure is known.

Find, correct to three places of decimals, the radius of a circle in which an arc 15 inches long subtends at the centre an angle containing $71^\circ 36' 3{\cdot}6''$. ($\pi = 3{\cdot}1416$.)

2. Define the *sine* of an angle, and prove that

$$\sin A = \sin (180^\circ - A) = \sin \{-(180^\circ + A)\}.$$

Write down formulæ including all angles which satisfy—

(1) $2 \sin A = 1$, (2) $2 \sin^2 A = 1$.

3. Prove that $\cos (A + B) = \cos A \cos B - \sin A \sin B$, and deduce expressions for $\cos 2A$, $\cos 3A$ in terms of $\cos A$.

4. Given $\cos A = {\cdot}28$, determine the value of $\tan \tfrac{1}{2} A$, and explain fully the reason of the ambiguity which presents itself in your result.

5. Prove that

(1) $\tan\theta + \cot\theta = \sqrt{\{\sec^2\theta + \operatorname{cosec}^2\theta\}}$.

(2) $\sec\theta - \tan\theta = \tan(\frac{1}{4}\pi - \frac{1}{2}\theta)$.

(3) $\cos 20^0 + \cos 100^0 + \cos 140^0 = 0$.

(4) $\cos^{-1}\frac{63}{65} + 2\tan^{-1}\frac{1}{5} = \sin^{-1}\frac{3}{5}$.

6. State and prove the rules by means of which you can determine by inspection the integral part of the logarithm of any given number.

Given $\log 4{\cdot}96 = {\cdot}6954817$, $\log 4{\cdot}9601 = {\cdot}6954904$, find the logarithms of 496010, ·000496, and 49600·25.

7. Shew that in *any* plane triangle $a = b\cos C + c\cos B$.

If $c = \sqrt{2}$, $A = 117^0$, $B = 45^0$, find all the other parts of the triangle.

8. Find the greatest angle of the triangle whose sides are 50, 60, 70 respectively, having given

$\log 6 = {\cdot}7781513$, $L\cos 39^0\,14' = 9{\cdot}8890644$, diff. $1' = 1032$.

9. Express the area of a triangle in terms of one side and the two angles adjacent to it.

Two angles of a triangular field are $22\frac{1}{2}^0$ and 45^0 respectively, and the length of the side opposite to the latter is a furlong. Shew that the field contains exactly two acres and a half.

10. Find an expression for the diameter of the circle which touches one side of a triangle and the other sides produced.

If d_1, d_2, d_3 be the diameters of the three escribed circles of a triangle, shew that $d_1d_2 + d_2d_3 + d_3d_1 = (a+b+c)^2$.

11. A man standing at a certain station on a straight sea-wall observes that the straight lines drawn from that station to two boats lying at anchor are each inclined at 45^0 to the direction of the wall, and when he walks 400 yards along the wall to another station he finds that the former angles of inclination are changed to 15^0 and 75^0 respectively. Find the distance between the boats, and the perpendicular distance of each from the sea-wall.

IV. University of Cambridge Local Examinations.

Junior. Dec. 1886.

1. Prove that the angle subtended at the centre of a circle by an arc equal to the radius is the same for all circles.

Express the angle as a fraction of a right angle.

2. Define the sine, cosine and tangent of an angle. Prove that these trigonometrical ratios are always the same for the same angle.

Find these ratios for an angle of 45°.

3. Prove that $\cos(A+B) = \cos A \cos B - \sin A \sin B$.

Prove that the sum of the cosines of two angles is equal to twice the cosine of half their sum multiplied by the cosine of half their difference.

4. Prove the following relations

(1) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

(2) $(1 + \sin A + \cos A)^2 = 2(1 + \sin A)(1 + \cos A)$

(3) $\dfrac{\sin 3A + \sin 5A}{\cos 3A - \cos 5A} = \cot A$.

5. Prove that the logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.

Having given $\log 2 = \cdot 3010300$, $\log 7 = \cdot 8450980$, find the logarithms of $(1\cdot 75)^{\frac{1}{2}}$, $(24\cdot 5)^{-\frac{1}{4}}$.

6. Prove that in any triangle ABC

(1) $a = b \cos C + c \cos B$.

(2) $2bc \cos A = b^2 + c^2 - a^2$.

7. Shew how to solve a triangle when one side and two angles are known.

Find the side b in the triangle ABC from the following data

$$a = 156\cdot 22,\quad B = 57°25',\quad C = 63°42',$$

$$\log 15\cdot 622 = 1\cdot 1937366,\quad L \sin 57°25' = 9\cdot 9256261,$$

$$\log 15\cdot 37552 = 1\cdot 1868297,\quad L \sin 58°53' = 9\cdot 9325330.$$

8. The angles of elevation of the top of a tower on a horizontal plane observed at two points distant a feet and b feet respectively from the base are found to be complementary. Shew that the height of the tower is $\sqrt{ab}$ feet. If θ be the angle subtended at the top of the lower by the line joining the two points, prove that

$$\sin \theta = \frac{a \sim b}{a + b}.$$

V. UNIVERSITY OF CAMBRIDGE LOCAL EXAMINATIONS.

Senior. Dec. 1886.

1. Explain the method of measuring angles in circular measure and find the circular measure of the angle of a regular pentagon.

Prove that $\dfrac{\tan^3 A}{1+\tan^2 A}+\dfrac{\cot^3 A}{1+\cot^2 A}=\dfrac{1-2\sin^2 A\cos^2 A}{\sin A\cos A}$.

2. Establish the identities:

(1) $\tan^4 A=\dfrac{3-4\cos 2A+\cos 4A}{3+4\cos 2A+\cos 4A}$.

(2) $\sin A-\sin B=2\sin\dfrac{A-B}{2}\cos\dfrac{A+B}{2}$.

If $A+B+C=180°$, shew that

$$(\sin B-\sin C)\cot\frac{A}{2}+(\sin C-\sin A)\cot\frac{B}{2}+(\sin A-\sin B)\cot\frac{C}{2}=0.$$

3. Find an expression for all the angles which have a given sine.

Solve the equations:

(1) $\cos\theta+\tan\theta=\sec\theta$. (2) $\sin\theta-2\sin 2\theta\cos\theta+\cos 3\theta=\cos 2\theta$.

4. Prove that in any triangle if a, b, c be the sides opposite to the angles A, B, C

(1) $\dfrac{\sin A}{a}=\dfrac{\sin B}{b}=\dfrac{\sin C}{c}$; (2) $\cos A+\cos B+\cos C=1+\dfrac{r}{R}$,

where r and R are the radii of the inscribed and circumscribed circles respectively.

If the line joining A to the centre of the inscribed circle meets the opposite side in D, prove that

$$\tan ADB=\frac{b+c}{b-c}\tan\frac{A}{2}.$$

5. If θ is the circular measure of a positive angle less than a right angle, shew that $\sin\theta$ lies between θ and $\theta-\dfrac{\theta^3}{6}$.

Find the limiting value of $\left(\dfrac{n}{\theta}\sin\dfrac{\theta}{n}\right)^n$ when n is indefinitely increased.

6. Prove that, if θ lie between $\dfrac{\pi}{4}$ and $-\dfrac{\pi}{4}$,

$$\theta=\tan\theta-\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta-$$

Shew that $\log(a+b\sqrt{-1})=\tfrac{1}{2}\log(a^2+b^2)+\sqrt{-1}\tan^{-1}\dfrac{b}{a}$.

VI. Mathematical Tripos, Part I. *June*, 1882.

1. Explain the different methods of measuring angles.

Find the number of degrees in each angle of a regular polygon of n sides (1) when it is convex, (2) when its periphery surrounds the inscribed circle m times.

Find correct to ·01 of an inch the length of the periphery of a decagon which surrounds an inscribed circle of a foot radius three times.

2. Prove geometrically the formula

$$\cos\alpha+\cos\beta=2\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\alpha-\beta)$$

Prove that

$$2\cos(\alpha-\beta)\cos(\theta+\alpha)\cos(\theta+\beta)+2\cos(\beta-\gamma)\cos(\theta+\beta)\cos(\theta+\gamma)$$
$$+\cos(\gamma-\alpha)\cos(\theta+\gamma)\cos(\theta+\alpha)-\cos 2(\theta+\alpha)-\cos 2(\theta+\beta)$$
$$-\cos 2(\theta+\gamma)-1$$

is independent of θ, and exhibit its value as a product of cosines.

3. Prove geometrically the formula

$$\tan(a+b)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.$$

Prove that if α, β, γ, δ be four solutions of the equation

$$\tan(\theta+\tfrac{1}{4}\pi)=3\tan 3\theta,$$

no two of which have equal tangents, then

$$\tan\alpha+\tan\beta+\tan\gamma+\tan\delta=0,$$

and

$$\tan 2\alpha+\tan 2\beta+\tan 2\gamma+\tan 2\delta=\tfrac{4}{3}.$$

4. Prove that in general the change in the cosine of an angle is approximately proportional to the change in the angle.

Prove that if in measuring the three sides of a triangle small errors x, y be made in two of them a, b, then the error in the angle C will be

$$-\left(\frac{x}{a}\cot B+\frac{y}{b}\cot A\right),$$

and find the errors in the other angles.

5. Prove that in any triangle $a \cos B + b \cos A = c$, and deduce the formula $$c^2 = a^2 + b^2 - 2ab \cos C.$$

Prove that if O be the centre of the circumscribing circle of the triangle ABC, the sides of the triangle formed by the centres of the three circles BOC, COA, AOB will be proportional to

$$\sin 2A \ : \ \sin 2B \ : \ \sin 2C.$$

Find the angles of the new triangle correct to one second when the sides of the triangle ABC are in the ratio of $4 : 5 : 7$.

6. Find the radius of the inscribed circle of a triangle in terms of one side and the angles.

Prove that if P be a point from which tangents to the three escribed circles of a triangle ABC are equal, the distance of P from the side BC will be

$$\tfrac{1}{2}(b+c) \sec \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C$$

VII. Oxford and Cambridge Schools Examination. *Eton*, 1883.

1. Given $\pi = 3{\cdot}1416$, find the number of degrees in the unit of circular measure of angles.

2. If $\sin\theta = \dfrac{2mn}{m^2+n^2}$, find $\tan\theta$, $\cos 2\theta$.

3. Prove that all angles included in the formula $2n\pi \pm \alpha$ have the same cosine as α.

Solve the equation $\cos\theta + \sqrt{3}\sin\theta = 2$.

4. Prove the equivalents:

(1) $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \operatorname{cosec}^2\theta$,

(2) $\cos\theta - \cos 3\theta = (\sin 3\theta - \sin\theta)\tan 2\theta$.

5. ABC is a triangle right-angled at A; BD meets AC in D: find AD in terms of CD and the angles ABC, ABD.

6. Shew that in any triangle

(1) $\cos \frac{1}{2}A = \sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$;

(2) $\tan \frac{1}{2}(B-C) = \frac{b-c}{b+c} \cot \frac{1}{2}A$,

(3) $(b-c)\cot \frac{1}{2}A + (c-a)\cot \frac{1}{2}B + (a-b)\cot \frac{1}{2}C = 0$.

7. Find an expression for the radius of the circumscribed circle of any triangle in terms of the sides.

The bisector of the angle A meets the side BC in D and the circumscribed circle in E, shew that $DE = \frac{a^2 \sec \frac{1}{2}A}{2(b+c)}$.

8. If the ratio of two sides of a triangle is $2 + \sqrt{3} : 1$ and the included angle is 60^0, find the other angles.

9. Shew that $\cos \frac{\theta}{n} + \sqrt{(-1)} \sin \frac{\theta}{n}$ is one of the values of

$$\{\cos \theta + \sqrt{(-1)} \sin \theta\}^{\frac{1}{n}},$$

n being a positive integer What are the other values?

10. Sum the series

(1) $\cos \theta + \cos 2\theta + \cos 3\theta + \ldots$ to n terms;

(2) $\sin \theta + \frac{1}{2}\sin 2\theta + \frac{1}{3}\sin 3\theta + \ldots$ to infinity

VIII. Oxford and Cambridge Schools Examination.

For Certificates, 1885

1. What is indicated by π?

What is meant by 'the angle π'?

The angles of a triangle are in A. P., and the ratio of the circular measure of the least to the number of degrees in the mean is 1 to 120. Find what multiple of a right angle is the greatest angle.

2. Define the tangent of an angle.

Prove that the tangent of a given angle has a single definite value. Find tan 30°, tan 90°.

3. Prove geometrically that

(1) $$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A};$$

(2) $$\tan\frac{A+B}{2} - \tan\frac{A-B}{2} = \frac{2\sin B}{\cos A + \cos B};$$

when A and B are each less than 90° and A is greater than B.

What is the relation between A, B, and C when

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1?$$

4. Prove the following statements:

(1) $\sin 11\theta \sin\theta + \sin 7\theta \sin 3\theta = \sin 8\theta \sin 4\theta$,

(2) $4\cos^8 A - 4\sin^8 A = 4\cos 2A - \sin 2A \sin 4A$;

(3) $\cos(x-y-z) + \cos(y-z-x) + \cos(z-x-y)$ $- 4\cos x \cos y \cos z$ vanishes, when $x+y+z$ is an odd multiple of a right angle.

5. Shew that, when two sides and the angle opposite one of them in a triangle are given, there may be two possible values for each of the other angles, or one, or the triangle may be impossible; distinguishing between the several cases.

Given that $A = 50°$, $b = 1071$, $a = 873$; and that

$$\log 1{\cdot}071 = {\cdot}029789, \qquad \log 8{\cdot}73 = {\cdot}941014,$$

$$\text{L}\sin 50° = 9{\cdot}884254, \qquad \text{L}\sin 70° = 9{\cdot}972986,$$

$$\text{L}\sin 70°\,1' = 9{\cdot}973032; \text{ find } B.$$

6. What is meant by $\sin^{-1} x$?

Prove that $\sin(\frac{1}{2}\sin^{-1} a)$ has four values for one value of a less than unity.

7. Prove that, if the inscribed circle of a triangle touch the sides opposite A, B, C in D, E, F respectively, the square of the area of the triangle is equal to the product of BD, CE, AF multiplied by their sum.

Prove that in any triangle with the usual notation

$$\frac{bc}{r_1}+\frac{ca}{r_2}+\frac{ab}{r_3}=2R\left\{\frac{b}{a}+\frac{c}{a}+\frac{c}{b}+\frac{a}{b}+\frac{a}{c}+\frac{b}{c}-3\right\}.$$

8. Write down the expansion of a^x in ascending powers of x, and deduce from it the theorem

$$\log_e(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\text{etc.}$$

Find the limit, when x is infinitely diminished, of

$$\frac{e^x-1+\log_e(1-x)}{\sin^3 x}.$$

9. Prove that, by giving to r the successive values 0, 1, 2, $(q-1)$, we obtain q different values of the expression

$$\cos\frac{a+2r\pi}{q}+\sqrt{(-1)}\sin\frac{a+2r\pi}{q},$$

and that, whatever integral value we may give to r, we cannot obtain more than q different values, q being any integer

10. Sum the series:

(1) $\sin 2a+\sin 5a+\sin 8a+$ etc. to n terms;

(2) $\sin 2a+n\sin 5a+\dfrac{n(n-1)}{1.2}\sin 8a+\ldots$ to $n+1$ terms.

IX. Oxford and Cambridge School Examinations.

For Certificates, 1886.

1. Explain the reasoning by which it is shewn that an angle may be measured in circular measure.

An observer who wants to determine the angle which the arc of a circle subtends at the centre makes an error of one-fifth of an inch in measuring the arc. The radius of the circle is 100 feet. What is the error in seconds in the angle so determined?

2. Define the tangent and cosecant of an angle; and express one in terms of the other.

Trace the changes in the sign and magnitude of $\tan A$ as A increases from 0° to 270°.

3. Find the value of $\sin 18^\circ$.

Points A, B, C, D are taken on the circumference of a circle so that the arcs AB, BC, and CD subtend respectively at the centre angles of 108°, 60°, and 36°. Shew that

$$AB = BC + CD.$$

4. Prove geometrically the following formulæ:

(1) $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$,

(2) $\tan\alpha + \tan\beta = (1 - \tan\alpha\tan\beta)\tan(\alpha+\beta)$.

5. Prove the following identities:

(1) $\sin 60^\circ = 4\sin 20^\circ \sin 40^\circ \sin 80^\circ$,

(2) $\sin(\alpha+\beta)\cos\beta - \sin(\alpha+\gamma)\cos\gamma = \sin(\beta-\gamma)\cos(\alpha+\beta+\gamma)$,

(3) $\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma)$
$$= 4\sin\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\gamma+\alpha)\sin\tfrac{1}{2}(\alpha+\beta).$$

6. Prove that the sides of a triangle are proportional to the sine of the opposite angles.

Three elements of a triangle are

$$a=2,\quad b=1,\quad A=48^\circ\ 20'.$$

Find the remaining angles, having given

$$\log 2 = 0\cdot30103,$$
$$\text{L}\sin 48^\circ\ 20' = 9\cdot8733352,$$
$$\text{L}\sin 21^\circ\ 55' = 9\cdot5720087,$$
$$\text{L}\sin 21^\circ\ 56' = 9\cdot5723226.$$

7. Find an expression for the radius, R, of the circles circumscribing a given triangle.

Shew that the radius of the circle inscribed in the triangle formed by joining the feet of the perpendiculars from the angles on the opposite sides is equal to

$$2R\cos A\cos B\cos C.$$

8. Shew that the height of a tower standing on a horizontal plane may be determined by observing its apparent altitude at any place in the plane whose distance from the base is known.

What place should be chosen for making the observation, so that if a small error be made in observing the altitude, the ratio of the error in the calculated height to the height may be as small as possible?

9. Prove that $\cos\frac{p}{q}\theta+\sqrt{-1}\sin\frac{p}{q}\theta$ is one of the values of

$$(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}.$$

Write down the other values.

Find all the sixth roots of 8.

10. Find the sum of n terms of the following series

(1) $\sin\theta+\sin(\theta+\alpha)+\sin(\theta+2\alpha)+$

(2) $\sin\theta+2\sin(\theta+\alpha)+3\sin(\theta+2\alpha)+$

X. Christ's Church, Oxford. Entrance Scholarships. 1883.

1. Prove geometrically that $\theta>\sin\theta>\theta-\frac{\theta^3}{4}$; θ being less than a right angle.

2. Prove the identities

(i) $\cot(A+B)\equiv\dfrac{\cot A\cot B-1}{\cot A+\cot B}$, geometrically.

(ii) $(\cos A+\sin A)(\cos 2A+\sin 2A)(\cos A-\sin 3A)$
$$\equiv\cos 2A\cos 4A.$$

(iii) $2^6(\cos^8 A+\sin^8 A)\equiv\cos 8A+28\cos 4A+35.$

(iv) $2\cos(n\cos^{-1}x)=(x+\sqrt{x^2-1})^n+(x-\sqrt{x^2-1})^n.$

3. Shew that

$$\log_e\frac{m}{n}=2\left\{\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\quad\right\}.$$

Having given that $\log_e 3=1{\cdot}0986$, find the value of $\log_{10}3$.

4. Eliminate α, β from the equations

$$x=(a\sin^2\alpha+b\cos^2\alpha)\cos^2\beta+c\sin^2\beta,\qquad y=a\cos^2\alpha+b\sin^2\alpha,$$
$$z=(b-a)\sin\alpha\cos\alpha\cos\beta.$$

5. If circles can be both described about, and inscribed in a quadrilateral, whose sides are a, b, c, d, and the angle between the diagonals θ, then

$$\theta=\cos^{-1}\frac{ac\sim bd}{ac+bd}.$$

6. Solve a triangle, having given the base a, altitude h, and the difference of the angles of the base α.

Account for the two values obtained for the vertical angle, and shew which of them is possible.

7. Shew that in a plane triangle

(i) $\Sigma(b-c)(s-a)\cos A=0$

(ii) $4R=r_a+r_b+r_c-r$.

(iii) $\dfrac{C}{2}=\dfrac{c}{a+b}\sin B+\frac{1}{2}\left(\dfrac{c}{a+b}\right)^2\sin 2B+\frac{1}{3}\left(\dfrac{c}{a+b}\right)^3\sin 3B+$

8. The triangle $A'B'C'$ circumscribes the escribed circles of the plane triangle ABC; shew that

$$\frac{B'C'}{a\cos A}=\frac{C'A'}{b\cos B}=\frac{A'B'}{c\cos C}.$$

9. If K be the centre of the nine-point circle of the triangle ABC, then $4AK^2=R^2+b^2+c^2-a^2$, where R is the radius of the circumscribing circle.

10. If $\cos(\theta+\phi\sqrt{-1})=\cos\alpha+\sqrt{-1}\sin\alpha$, and α, θ, ϕ are real, prove that $\tan^2\theta-\tan^2\alpha=\sin^2\theta\sec^2\alpha$, and find a relation between θ and ϕ.

11. Sum to infinity the series—

(i) $\cos\alpha\tan\phi-\frac{1}{3}\cos 3\alpha\tan^3\phi+\frac{1}{5}\cos 5\alpha\tan^5\phi$

(ii) $(1-3^{-\frac{1}{2}})-\frac{1}{3}(1-3^{-\frac{3}{2}})+\frac{1}{5}(1-3^{-\frac{5}{2}})$

(iii) $\dfrac{1}{\pi^2-x^2}+\dfrac{1}{4\pi^2-x^2}+\dfrac{1}{9\pi^2-x^2}+$

XI. St John's College, Cambridge. *June Exam.*, 1879.

1. Explain the method of measuring angles by degrees, minutes, &c.

The numerical measures of the angles A, B, C of a triangle when referred to units l^0, m^0, n^0, respectively, are in arithmetical progression, and when referred to units p^0, q^0, r^0 respectively, they are in geometrical progression. Find A, B, C.

2. Define the sine and cosine of an angle, and prove that

$$\sin^2 A + \cos^2 A = 1$$

If $\cos^2 A + \cos B = 1 = \sin^2 A + \sin B$, find A and B.

3. Prove geometrically that

(1) $\sin 2A = 2 \sin A \cos A$,

(2) $\tan A = \dfrac{\sin 2A}{1 + \cos 2A}$.

If $\theta = \dfrac{\pi}{2^n + 1}$, prove that

$$2^n \cos\theta \cos 2\theta \cos 2^2\theta \ldots \cos 2^{n-1}\theta = 1.$$

4. Prove that

$$\cos A + \cos B = 2 \cos \tfrac{1}{2}(A + B) \cos \tfrac{1}{2}(A - B)$$

Find θ and ϕ from the equations

$$\cos\alpha \{\cos\alpha + \cos(\alpha + \theta)\} = \cos\beta \{\cos\beta + \cos(\beta + \phi)\},$$

$$\cos\alpha \{\sin\alpha + \sin(\alpha + \theta)\} = \cos\beta \{\sin\beta + \sin(\beta + \phi)\}.$$

5. If θ be the circular measure of an angle less than a right angle, prove that $\sin\theta$, θ, and $\tan\theta$, are in ascending order of magnitude.

If the unit of measurement be a right angle, find the limit of $\dfrac{\tan\theta - \sin\theta}{\theta^3}$ as θ is indefinitely diminished.

6. Expand $\log_a (1+x)$ in a series of powers of x

Prove that

$$2\left(\cos A + \tfrac{1}{3}\cos^3 A + \tfrac{1}{5}\cos^5 A \qquad\right) = \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A + \tfrac{1}{2}(\cos^4 \tfrac{1}{2}A - \sin^4 \tfrac{1}{2}A)$$
$$+ \tfrac{1}{3}(\cos^6 \tfrac{1}{2}A - \sin^6 \tfrac{1}{2}A) + \dots$$

7. In any triangle the sides are proportional to the sines of the angles opposite to them.

Through the angular point C of a triangle ABC is drawn any line CMN on which are dropped perpendiculars AM, BN. Prove that

$$MN = AN \cot B \sim BN \cot A.$$

8 Express the sine and cosine of half the angle of a triangle in terms of the sides.

If ABC, $A'B'C'$ be two triangles, such that

$$\frac{1+\cos A}{a'} = \frac{1+\cos B}{b'} = \frac{1+\cos C}{c'},$$

prove that $\tan \frac{1}{2}A \tan \frac{1}{2}A' = \tan \frac{1}{2}B \tan \frac{1}{2}B' = \tan \frac{1}{2}C \tan \frac{1}{2}C'$.

9 Give the formulæ for the solution of a triangle in which one angle and the containing sides are given.

If $C = 44^0$, $a = 43$ ft, $b = 11$ ft, find A and B

Having given

$\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$, $L \tan 22^0 = 9{\cdot}6064096$,

$L \tan 34^0\, 17' = 9{\cdot}8336109$, $L \tan 34^0\, 19' = 9{\cdot}8338823$.

10 Enunciate and prove De Moivre's theorem

If n be equal to $3m \pm 1$, prove that

$$(1 \mp n) - \left\{\frac{n(n-1)}{\underline{|2}} \mp \frac{n(n-1)(n-2)}{\underline{|3}}\right\} 3$$
$$+ \left\{\frac{n(n-1)(n-2)(n-3)}{\underline{|4}} \mp \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}}\right\} 3^2 - \&c. = 0.$$

11 Find the sum of the following series, each to n terms.

(1) $\cos a + \cos 3a + \cos 5a + \quad .$

(2) $\tan^{-1} 2 + \tan^{-1} \dfrac{4}{1+3 \,.\, 4} + \tan^{-1} \dfrac{6}{1+8 \,.\, 9} + \tan^{-1} \dfrac{8}{1+15 \,.\, 16} +$

12. Resolve $x^{2n} - 2x^n \cos\theta + 1$ into factors.

If n be an even integer, prove that

$$\sin^2\frac{\theta}{2} = 2^{n-2}(-1)^{\frac{n}{2}} \cos\frac{\theta}{n} \cos\frac{2\pi+\theta}{n} \cos\frac{4\pi+\theta}{n} \dots \cos\frac{(2n-2)\pi+\theta}{n}.$$

XII. St John's College, Cambridge. Minor Scholarship, 1881.

1. Shew that in the expression for $\tan\frac{A}{2}$ in terms of $\tan A$ we should *à priori* expect a double result. Find $\tan 112^\circ.30'$.

2. A triangle is such that the product of two sides is equal to the square on half the base: prove that the difference of the sides varies as the distance from the vertex to the middle point of the base.

3. (i) If x, y, z be any angles, prove that

$$\sin\tfrac{1}{2}(x-y-z)\sin\tfrac{1}{2}(y-z) + \sin\tfrac{1}{2}(x+y-z)\sin\tfrac{1}{2}(y+z) = \sin\tfrac{1}{2}x\sin y.$$

(ii) Also if A, B, B be the angles of an isosceles triangle,

$$2\sin^2(A-B)(2-\cos A) = (\sin^2 A + 2\sin^2 B)(1 - 8\cos A\cos^2 B).$$

4. (i) Eliminate θ from

$$\left.\begin{array}{l} 2\cos^2\theta + \lambda\sec\theta = 3 \\ 2\sin^2\theta + \mu\,\mathrm{cosec}\,\theta = 3 \end{array}\right\}.$$

(ii) If $x^2\cos\alpha\cos\beta + x(\sin\alpha + \sin\beta) + 1 = 0$

and $x^2\cos\beta\cos\gamma + x(\sin\beta + \sin\gamma) + 1 = 0$,

prove that $x^2\cos\gamma\cos\alpha + x(\sin\gamma + \sin\alpha) + 1 = 0$.

5. Prove that the distance between the centre of the inscribed circle and the intersection of perpendiculars from the angular points on the opposite sides of a triangle is

$$2R\left\{\text{vers}\,A\,\text{vers}\,B\,\text{vers}\,C - \cos A\cos B\cos C\right\}^{\frac{1}{2}},$$

where R is the radius of the circumscribed circle.

6. Prove that $\{\cos\theta + \sqrt{(-1)}\sin\theta\}^{\frac{p}{q}}$ admits of no more than q values.

Find the continued product of the 4 values of

$$\{\cos\tfrac{1}{3}\pi + \sqrt{(-1)}\sin\tfrac{1}{3}\pi\}^{\frac{3}{4}}.$$

XIII. Examination for Open Scholarships at Gonville and Caius College. *Dec.*, 1886.

(*The same paper was also set for Open Scholarships at Pembroke College.*)

1. Define the cosine of any angle, and prove that

$$\cos\left(\frac{\pi}{2}+A\right)=-\sin A.$$

If $\cos 6A$ be given, how many different values are possible for $\sin A$?

Solve the equation

$$\sin 7\theta+\sin 5\theta-\cos 2\theta=1.$$

2. Prove geometrically that

$$\cos(A-B)=\cos A\cos B+\sin A\sin B,$$

where A and B are each less than a right angle.

Shew that
$$\cos^{-1}\tfrac{4}{5}-\sin^{-1}\frac{1}{\sqrt{10}}+\tan^{-1}\tfrac{1}{2}=\frac{\pi}{4},$$

where the angles denoted by the inverse functions are supposed to lie between 0 and $\dfrac{\pi}{2}$.

3. If $A+B+C=\pi$, prove that

(1) $\cos A+\cos B+\cos C=1+4\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$,

(2) $$\sin^2\left(\frac{A}{2}+B\right)+\sin^2\left(\frac{B}{2}+C\right)+\sin^2\left(\frac{C}{2}+A\right)$$
$$=1+2\sin\left(\frac{A}{2}+B\right)\sin\left(\frac{B}{2}+C\right)\sin\left(\frac{C}{2}+A\right).$$

4. Prove that, in any triangle ABC,

$$a^2=b^2+c^2-2bc\cos A.$$

If $C=\dfrac{\pi}{3}$ shew that

$$2\cos(A-B)=\frac{4ab-a^2-b^2}{a^2-ab+b^2}.$$

5. State the values of the radii of the circumscribed and escribed circles of a triangle ABC. If R be the radius of the circumscribed circle, and r_1 the radius of that escribed circle which touches BC, prove that

$$r_1 = 4R \sin\frac{A}{2} \cos\frac{B}{2} \cos\frac{C}{2}.$$

If D be the centre of the inscribed circle, and E the centre of that escribed circle which touches BC, prove that the area of the quadrilateral $DBEC$ is

$$4R^2 \sin^2\frac{A}{2} (\sin B + \sin C).$$

6. Assuming the exponential value for the tangent of an angle, prove that, under certain restrictions,

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \quad .$$

If both θ and $\tan^{-1}(\sec\theta)$ lie between 0 and $\frac{\pi}{2}$, prove that

$$\tan^{-1}(\sec\theta) = \frac{\pi}{4} + \tan^2\frac{\theta}{2} - \tfrac{1}{3}\tan^6\frac{\theta}{2} + \tfrac{1}{5}\tan^{10}\frac{\theta}{2} -$$

vii. Shew that if n is integral, $\sin 2n\theta$ and $\cos 2n\theta$ can be expressed as rational algebraical functions of $\tan\theta$.

Express $\sin 10\theta$ in this way, and shew how to deduce the values of the tangents of $\frac{n\pi}{10}$ from $n=1$ to $n=9$.

viii. Defining the functions $\sinh\theta$, $\cosh\theta$ by the equations

$$\sinh\theta = \tfrac{1}{2}(e^{\theta} - e^{-\theta}),\ \cosh\theta = \tfrac{1}{2}(e^{\theta} + e^{-\theta}),$$

prove the formula

$$\sinh(\alpha \pm \beta) = \sinh\alpha \cosh\beta \pm \cosh\alpha \sinh\beta$$

$$\sin(\alpha \pm \beta\iota) = \sin\alpha \cosh\beta \pm \iota \cos\alpha \sinh\beta.$$

Shew that the limit of $\cosh\theta \cosh\frac{\theta}{2} \quad \cosh\left(\frac{\theta}{2}\right)^n$, when n is indefinitely increased, is $\frac{\sinh 2\theta}{2\theta}$.

XIV. CLARE, CAIUS, AND KING'S COLLEGES. *June Exam.*, 1880.

1. Draw a curve representing the change in sign and magnitude of $\tan 2\theta$ while θ changes from 0 to π

Do the same for $\tan 2\theta - 2\tan\theta$.

2. Prove geometrically

$$(1)\quad \tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B},$$

$$(2)\quad \cos 3A = 4\cos^3 A - 3\cos A.$$

If $3(1+\tan^2 A\tan^2 B) + 8\tan A\tan B = \tan^2 A + \tan^2 B$, A and B differ by some multiple of $\frac{1}{3}\pi$.

3. If $\sin 3A$ be given, and from this value $\tan A$ is to be found, shew *à priori* that six values are to be generally expected

Prove by help of this, or otherwise, that

$$\tan^2\alpha\,\{\tan^2(\tfrac{1}{3}\pi - \alpha) + \tan^2(\tfrac{1}{3}\pi + \alpha)\} + \tan^2(\tfrac{1}{3}\pi - \alpha)\tan^2(\tfrac{1}{3}\pi + \alpha)$$
$$= 6\sec^2 3\alpha + 3.$$

4 If $\theta+\phi+\psi=0$, prove that

$$\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\phi\tan\tfrac{1}{2}\psi = -\frac{\sin\theta+\sin\phi+\sin\psi}{1+\cos\theta+\cos\phi+\cos\psi} \qquad (1)$$

Find $\cos x$ from the equation

$$\{4\cos(x+\alpha)-1\}\,\{4\cos(x-\alpha)-1\} = 5\,(2\cos 2\alpha - 1) \qquad (2).$$

Eliminate α from the equations

$$\frac{\sin\theta}{\beta^2-1} = \frac{\cos\theta}{2\beta\sin 2\alpha} = \frac{1}{1+2\beta\cos 2\alpha+\beta^2},$$

shewing that $\qquad \beta = \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta) \qquad (3).$

5. State the principle of proportional parts in the use of tables of functions. What is meant by saying that the differences are (1) insensible, (2) irregular?

Prove that they are both insensible and irregular in the case of the logarithmic sine when the angle approaches $\frac{1}{2}\pi$.

Determine a limit to the error which can be made in finding the logarithm of $N+\dfrac{a}{100}$ from seven-figure tables from those of N and $N+1$, where a lies between 0 and 100 and N consists of 5 digits.

6. Explain fully the method of solving a triangle, given two sides, the included angle and a table of logarithms.

ABC, $AB'C'$ are two triangles having AB, BC equal respectively to AB', $B'C'$, and A, C, C' are collinear. If the angle BAB' is $1''$, find correctly to a tenth of a second the angle between BC and $B'C'$, where $AB=2BC$ and $\angle ABC=60^\circ$.

7. ABC is a triangle and tangents are drawn to the nine-point and circumscribing circles at the four points where the perpendicular from A on the opposite side BC meets them.

Prove that the four tangents form a parallelogram of area

$$R^2 \frac{\cos A \cos B \cos C}{\tan (B \sim C)}.$$

8. Find the limit of the expression $(90-\theta) \tan \theta^\circ$ as θ approaches 90.

9. Given that $x^m + \frac{1}{x^m} = 2 \cos m\theta$ for all values from 0 to m, shew that the formula holds when $m+1$ is written for m. Deduce the value of $x^{\frac{r}{s}} + \frac{1}{x^{\frac{r}{s}}}$, in the most general form.

10. Prove that $\qquad \sin \theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}}$

and deduce the exponential value of $\sin \theta$.

Shew that $\sin^{-1} (\operatorname{cosec} \theta) = (2\lambda + 1) \frac{1}{2}\pi + \sqrt{(-1)}\, l_e \cot \frac{1}{2} (\lambda\pi + \theta)$, where λ is any integer positive or negative.

11. Assuming the factorial expressions for $\sin \theta$ and $\cos \theta$, prove that $\tan \theta > \theta$, provided θ lie between 0 and $\frac{1}{2}\pi$.

By means of the result in question 8, or otherwise, prove that the infinite product

$$\frac{4 \cdot 5}{2 \cdot 4} \cdot \frac{5 \cdot 6}{4 \cdot 6} \cdot \frac{6 \cdot 7}{6 \cdot 8} \cdot \frac{7 \cdot 8}{8 \cdot 10} \ldots \text{ is equal to } \frac{16}{3\pi}.$$

12. Sum the series,

$$\cos \alpha + \cos (\alpha + \beta) + \ldots \text{ to } n \text{ terms} \qquad (1),$$

$$\sin \alpha + 3 \sin 2\alpha + 5 \sin 3\alpha + \ldots \text{ to } n \text{ terms} \qquad (2).$$

XV. Christ's, Emmanuel, and Sidney Sussex Colleges.

June Examination, 1882.

1. Define the cosine and the tangent of an angle.

Trace the changes in sign and magnitude of

$$(1)\quad \tan\theta, \qquad\qquad (2)\quad \frac{\sqrt{3}+\tan\theta}{\sqrt{3}-\tan\theta},$$

as θ varies from 0^0 to 360^0.

2. Prove geometrically that

$$\sin A - \sin B = 2\sin\tfrac{1}{2}(A-B)\sin\tfrac{1}{2}(A+B).$$

Shew that

$$(1)\quad \cos 2\alpha\cos^2(\beta+\gamma)+\cos 2\beta\cos^2(\gamma+\alpha)+\cos 2\gamma\cos^2(\alpha+\beta)$$
$$=\cos 2\alpha\cos 2\beta\cos 2\gamma+2\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta),$$

$$\frac{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha+\gamma)}{\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\alpha-\gamma)}\cos\alpha+\frac{\sin\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}(\beta+\alpha)}{\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\beta-\alpha)}\cos\beta$$
$$+\frac{\sin\frac{1}{2}(\gamma+\alpha)\sin\frac{1}{2}(\gamma+\beta)}{\sin\frac{1}{2}(\gamma-\alpha)\sin\frac{1}{2}(\gamma-\beta)}\cos\gamma=\cos(\alpha+\beta+\gamma).$$

3. Find the limit of $\dfrac{\sin\theta}{\theta}$ when θ is diminished indefinitely.

In order to ascertain the distance of an inaccessible object P, a person measures a length $AB=20$ yards in a convenient direction; at A he observes that the angle $PAB=60^0$, and at B that the angle $PBA=119^0\ 20'$. Find approximately the distance BP. To what degree of accuracy is your result correct, supposing (1) that there is no error in the measurement of the angles, (2) that there is an error of $1'$ in the measurement of each angle?

4. In any triangle ABC, shew that

$$a=b\cos C+c\cos B,$$

and that
$$a^2=b^2+c^2-2bc\cos A.$$

If N be the foot of the perpendicular from C on AB, and the circle on CN as diameter cut CA, CB in P and Q respectively, shew that the angle BPN is equal to the angle AQN.

5. Express the area of a triangle in terms of its sides

A straight line AB is divided at C into two parts of lengths $2a$ and $2b$ respectively. On AC, CB and AB as diameters semicircles are described so as to be on the same side of AB. If O be the centre of the circle which touches each of the three semicircles, shew that its radius

$$=\frac{ab\,(a+b)}{a^2+ab+b^2},$$

and that its diameter is equal to the altitude of the triangle AOB

6. Shew how to find the height and distance of an inaccessible object on a horizontal plane.

A person wishing to ascertain the height of a tower stations himself in a horizontal plane through the base at a point at which the elevation at the top is 30^0. On walking a distance a in a certain direction he finds that the elevation of the top is the same as before, and on walking a distance five-thirds of a at right angles to his previous direction, he finds that the elevation of the top is 60^0. Shew that the height of the tower is $\sqrt{\frac{5}{6}}a$ or $\sqrt{\frac{25}{48}}a$ Explain the two results.

7. In a triangle ABC, I, I' and O are the centres of the inscribed circle, the escribed circle opposite A and the circumscribing circle respectively, and R is the radius of the latter circle. Shew that

$$(1)\quad OI^2=R^2\,(1-8\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C),$$

$$(2)\quad \tan IOI'=\pm\frac{2\,(\sin B-\sin C)}{2\cos A-1}.$$

8. Explain the meanings of $\sin^{-1}x$ and $\tan^{-1}x$.

How many bounding lines are required to construct all the angles included in the formula

$$\sin^{-1}a+\cos^{-1}b+\tan^{-1}c\,?$$

Shew that $\sin^{-1}a+\cos^{-1}b=\sin^{-1}(ab+\sqrt{1-a^2}\sqrt{1-b^2})$.

If $xy+yz+zx=1$, prove that one of the values of

$$\sin^{-1}\frac{2\,(x+y)\,(1-xy)}{(1-x^2)\,(1+y^2)}+\sin^{-1}\frac{(y+z)\,(1-yz)}{(1+y^2)\,(1+z^2)}+\sin^{-1}\frac{(z+x)\,(1-2x)}{(1+z^2)\,(1+x^2)}=2\pi.$$

9. Assuming De Moivre's Theorem find the expansions of $\sin n\theta$ and $\cos n\theta$ as homogeneous functions of $\sin\theta$ and $\cos\theta$.

Find the equation whose roots are $\tan^2\frac{\pi}{11}$, $\tan^2\frac{2\pi}{11}$, $\tan^2\frac{3\pi}{11}$, $\tan^2\frac{4\pi}{11}$, and $\tan^2\frac{5\pi}{11}$. Find also the sum of the fourth powers of these tangents.

10. Investigate Gregory's series for the expansion of $\tan^{-1}x$ in powers of x.

Expand $\tan^{-1}(x+\cot\alpha)$ in powers of x.

11. Prove that

$$\cos n\theta = 2^{n-1}\left(\cos\theta - \cos\frac{\pi}{2n}\right)\left(\cos\theta - \cos\frac{3\pi}{2n}\right)\ldots\left(\cos\theta - \frac{(2n-1)\pi}{2n}\right)$$

Shew also that if $p < n$

$$\frac{\cos^p\theta}{\cos n\theta} = \frac{1}{n}\sum_{r=0}^{r=n-1}(-1)^r\frac{\sin\frac{(2r+1)\pi}{2n}\cos^p\frac{(2r+1)\pi}{2n}}{\cos\theta-\cos\frac{(2r+1)\pi}{2n}}$$

12. If $A, B, C, D \ldots$ be the angular points of a regular polygon of n sides inscribed in a circle of radius a and centre O, shew that

$$PA^2 \,.\, PB^2 \,.\, PC^2 \ldots = r^{2n} - 2a^n r^n \cos n\theta + a^{2n},$$

where $OP=r$ and the angle $AOP=\theta$.

Prove also that the sum of the angles that AP, BP, CP, ... make with OP is $\tan^{-1}\dfrac{r^n\sin n\theta}{r^n\cos n\theta - a^n}$.

ANSWERS TO THE EXAMPLES.

I

$$\begin{array}{rll}
 & 1 & = 1 \\
2) & 1 & = 1 \\
3) & \cdot 5 & = \frac{1}{2} \\
4) & \cdot 1666666 & = \frac{1}{2}\times\frac{1}{3} \\
5) & \cdot 0416666 & = \frac{1}{2}\times\frac{1}{3}\times\frac{1}{4} \\
6) & \cdot 0083333 & = \frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5} \\
7) & \cdot 0013888 & = \frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}\times\frac{1}{6} \\
8) & \cdot 0001984 & = \frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}\times\frac{1}{6}\times\frac{1}{7} \\
9) & \cdot 0000248 & = \frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}\times\frac{1}{6}\times\frac{1}{7}\times\frac{1}{8} \\
10) & \cdot 0000027 & = \frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}\times\frac{1}{6}\times\frac{1}{7}\times\frac{1}{8}\times\frac{1}{9} \\
 & \cdot 0000002 & = \frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times\frac{1}{5}\times\frac{1}{6}\times\frac{1}{7}\times\frac{1}{8}\times\frac{1}{9}\times\frac{1}{10}
\end{array}$$

Whence by addition $\left.\right\}$ $\underline{2\cdot 71828} = 1+1+\frac{1}{2}+\frac{1}{1\,.\,2\,.\,3}+\frac{1}{1\,.\,2\,.\,3\,.\,4}+$

$$+\frac{1}{1\,.\,2\,.\,3\,.\,4\,.\,5\,.\,6\,.\,7\,.\,8\,.\,9\,.\,10}+ \;.$$

II.

(3) The student should observe that each of these series is very slowly convergent.

III.

(2) $\log_e 7 = 1\cdot 9459$.

V.

Examples (1), (2), (3) indicate a method of obtaining the Logarithmic and Exponential Expansions.

(6) This is a form of Euler's Formula

$$\sin\theta = 2^n \cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3} \quad \cos\frac{\theta}{2^n}\sin\frac{\theta}{2^n}$$

when θ is indefinitely increased

VI.

(12) (i) $(-1)^{\frac{n-2}{2}} s_{n-1}$ (ii) $(-1)^{\frac{n-1}{2}} s_n$

VII.

(1) $\sqrt{2}\,\{\cos(-45^0)+\iota\sin(-45^0)\}$, $2(\cos 30^0+\iota\sin 30^0)$, $2(\cos 60^0+\iota\sin 60^0)$.

(2) (i) $2(\cos 15^0+\iota\sin 15^0)$, $2(\cos 135^0+\iota\sin 135^0)$, $2(\cos 255^0+i\sin 255^0)$.

(ii) $2(\cos 10^0+\iota\sin 10^0)$, $2(\cos 130^0+\iota\sin 130^0)$, $2(\cos 250^0+\iota\sin 250^0)$.

(iii) $\sqrt[5]{2}(\cos 6^0+\iota\sin 6^0)$, $\sqrt[5]{2}(\cos 78^0+\iota\sin 78^0)$, $\sqrt[5]{2}(\cos 150^0+\iota\sin 150^0)$, $\sqrt[5]{2}(\cos 222^0+\iota\sin 222^0)$, $\sqrt[5]{2}(\cos 294^0+\iota\sin 294^0)$.

(3) (i) ± 1, $\pm\sqrt{(-1)}$.

(ii) 2, $2\{\cos\frac{1}{5}(2r\pi)+\iota\sin\frac{1}{5}(2r\pi)\}$ putting 1, 2, 3, 4 for r successively.

(iii) 3, $\frac{3}{2}\{-1+\iota\sqrt{3}\}$, $\frac{3}{2}\{-1-\iota\sqrt{3}\}$.

VIII.

(1) 1, $\cos\frac{1}{5}(2r\pi)+\iota\sin\frac{1}{5}(2r\pi)$, where $r=1, 2, 3, 4$

(2) $(x^2-1)(x^2-\sqrt{2}x+1)(x^2+1)(x^2+\sqrt{2}x+1)$.

(3) $(x-1)\{x^2-2x\cos\frac{1}{13}(2r\pi)+1\}$ six factors putting 1, 2, 3, 4, 5, 6 for r.

(4) $(x^2-1)(x^2-x+1)(x^2+x+1)$. (5) $x^2-2ax\cos\frac{1}{15}(r\pi)+a^2$.

(6) $\cos\frac{1}{6}(r\pi)+\iota\sin\frac{1}{6}(r\pi)$, r having each integral value from 0 up to 11.

IX.

(1) $\pm\frac{1}{\sqrt{2}}\pm\sqrt{(-1)}\frac{1}{\sqrt{2}}$, $(x^2-x\sqrt{2}+1)(x^2+x\sqrt{2}+1)$.

(2) $(x^2-\sqrt{3}x+1)(x^2+1)(x^2+\sqrt{3}x+1)$.

(3) $x^2-2x\cos(1+2r)\,9^0+1$.

(4) Solve the equation $(x^2-\sqrt{3}x+1)(x^2+1)(x^2+\sqrt{3}x+1)=0$.

(5) $(x+1)\{x^2-2x\cos\frac{1}{13}(\pi+2r\pi)+1\}$ seven factors in all.

(6) $\cos\frac{\pi+2r\pi}{n}+\iota\sin\frac{\pi+2r\pi}{n}$, where r may have any integral value.

X.

(3) With the notation of Ex. (1)

$$\sin(\alpha+\beta+\gamma+\delta+\epsilon)=S_1C_4-S_3C_2+S_5$$
$$\cos(\alpha+\beta+\gamma+\delta+\epsilon)=C_5-S_2C_3+S_4C_1.$$

XIII

(9) Put $b=0$ in Ex. 3, p. 28.

(10) Put $\alpha=0$, $\beta=1$ in Ex 3, p 28

(17) $\sqrt[3]{i}=\cos\dfrac{2r\pi+\pi}{6}+i\sin\dfrac{2r\pi+\pi}{6}$; making this substitution, $a^{\sqrt[3]{i}}$ is expressed in the form a^{a+ib}. Then proceed as in Ex. 3, p. 28.

(18) $$c^i=e^{i\log c}=\cos(\log c)+i\sin(\log c),$$
$$\therefore\ a+ib+c^i=\{a+\cos(\log c)\}+i\{b+\sin(\log c)\}$$

Then proceed as in Ex 3, p 28.

XIV.

(1) The equation $e^{6i\pi}=e^0$, does not assert that $6i\pi=0$, but that $6i\pi=0+2ir\pi$.

(2) $e^{i(\alpha-\pi)}=e^{i(\alpha+\pi)}$, does not assert that $i(\alpha-\pi)=i(\alpha+\pi)$.

XVII.

(1) $(x^2-2x\cos 15^\circ+1)(x^2-2x\cos 105^\circ+1)(x^2-2\cos 195^\circ+1)$
$(x^2-2x\cos 285^\circ+1)=0$

(2) $(x^2-2x\cos 2^\circ+1)(x^2-2x\cos 74^\circ+1)(x^2-2x\cos 146^\circ+1)$
$\times(x^2-2x\cos 218^\circ+1)(x^2-2x\cos 290^\circ+1)=0.$

(3) $x^2-2x\cos\frac{1}{9}(3r\pi+\pi)+1=0$, six quadratics.

(4) $x^2-2x\cos(r\times 72^\circ+30^\circ)+1=0$, five quadratics.

(5) $x^2-2xy\cos\dfrac{a+2r\pi}{n}+y^2$, n factors.

XIX.

(5) θ is the smaller root of the quadratic

$$\theta^2-2\theta+(\cdot 0029\ldots)2\sqrt{2}=0.$$

(7) $2m^{-2}$. (8) $\dfrac{2n^2-2m^2}{p^2}$. (9) $\dfrac{2(m-n)^2}{3mn}$.

(10) $\frac{1}{60}$.

XXII.

(4) (i) $\dfrac{2n(2n-1)\,.\,.\,(n+1)}{n(n-1)\ldots 2\,.\,1}$,

(ii) $\dfrac{(2n+1)\,2n\,(2n-1)\quad(n+2)}{n\,.\,(n-1)\,.\,.\,2\,.\,1}\,2\cos\theta$

(iii) $-\dfrac{(4n+2)(4n+1)\quad(2n+2)}{(2n+1)\,2n\,(2n-1)\ldots 2\,.\,1}$.

XXV.

(1) n even, $\{(-1)^{\frac{n}{2}}-\cos n\alpha\}\,2^{-n+1}$, n odd, $2^{-n+1}\cos n\alpha$

(2) n even, 0, n odd $(-1)^{\frac{n-1}{2}}\dfrac{n}{\cos n\alpha}$.

(3) n even, $(-1)^{\frac{n}{2}}(1-\cos n\alpha)\,2^{-n+1}$, n odd $(-1)^{\frac{n-1}{2}}\,2^{-n+1}\sin n\alpha$

(4) n even, $\dfrac{n^2}{1-\cos n\alpha}$, n odd $\dfrac{n^2}{\sin^2 n\alpha}$.

(5) (6) (7) $$\tan n\alpha=\frac{n\tan\theta-\dfrac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\tan^3\theta+}{1-\dfrac{n(n-1)}{1\,.\,2}\tan^2\theta+\text{etc.}}$$

by Art. 40 is an equation of the n^{th} degree in $\tan\theta$, of which $\tan\alpha$ is one root, and $\tan\left(\alpha+\dfrac{r\pi}{n}\right)$ is another; then proceed as in Art. 67

(5) The sum, n even, $=\dfrac{n^2}{\tan^2 n\alpha}+n(n-1)$,

n odd $=n^2\tan^2 n\alpha+n(n-1)$.

(6) The sum, n even, $=-\,n\cot n\alpha$; n odd, $=n\tan n\alpha$.

(7) The sum $=n\cot n\alpha$ when n is even and $=-n\tan n\alpha$ when n is odd.

XXVI.

(3) Let $u=(\cos\theta)^{\cot\theta}$, then $\log u=\cot\theta\log(\cos\theta)$

$$=(1-\tfrac{1}{2}\theta^2+\quad)(-\tfrac{1}{2}\theta^2+\quad)\div(\theta-\tfrac{1}{6}\theta^3+\quad)$$

the limit of this $=0$ when $\theta=0$, therefore the limit of u is 1

(4) $-\infty$. (5) Of the fifth order.

(8) $2^{4n+2}\left\{\cos 4n\theta + \frac{4n(4n-1)}{1\,.\,2}\cos 4(n-1)\theta + \frac{4n(4n-1)(4n-2)(4n-3)}{1\,.\,2\,.\,3\,.\,4}\cos 4(n-2)\theta + \quad\right\}.$

(10) By Art. 40, or Ex. 12, p. 47, the equation

$$\tan n\alpha = \frac{n\tan\theta - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}\tan^3\theta + }{1 - \frac{n(n-1)}{1\,.\,2}\tan^2\theta + \,..}$$

has for its roots α, $\alpha + \frac{\pi}{n}$, $\alpha + \frac{2\pi}{n}$,

Put $\alpha = 0$, $n = 11$, and divide by $\tan\theta$, when we have an equation in $\tan\theta$, viz.

$$0 = 1 - 15\tan^2\theta + 42\tan^4\theta - 30\tan^6\theta + 5\tan^8\theta - \tfrac{1}{11}\tan^{10}\theta,$$

whose roots are $\pm\tan\frac{1}{11}\pi$, $\pm\tan\frac{2}{11}\pi$, $\quad \pm\tan\frac{5}{11}\pi$, writing x for $\tan^2\theta$, we have the required equation

XXVII.

(1) $\dfrac{\sin\frac{1}{2}(n+1)\alpha\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}$. (2) $\dfrac{\cos n\alpha\,.\,\sin n\alpha}{\sin\alpha}$.

(3) $\dfrac{\sin\frac{1}{2}(3n-1)\alpha\sin\frac{3}{2}n\alpha}{\sin\frac{3}{2}\alpha}$. (4) $\dfrac{\sin(n+1)\alpha\sin n\alpha}{2\sin\alpha}$

(5) $\frac{1}{2}\left\{n + \dfrac{\cos(n+1)\alpha\sin n\alpha}{\sin\alpha}\right\}$. (6) Use $\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$.

(7) Use $\cos^4\alpha = \frac{1}{8}\cos 4\alpha + \frac{1}{2}\cos 2\alpha + \frac{3}{8}$.

(8) $= \frac{1}{2}(\sin 3\alpha + \sin\alpha) + \frac{1}{2}(\sin 5\alpha + \sin\alpha) + \frac{1}{2}(\sin 7\alpha + \sin\alpha) +$

(9) $= \frac{1}{2}\{\cos\alpha - \cos 3\alpha + \cos\alpha - \cos 5\alpha + \cos\alpha - \cos 7\alpha + \quad\}$

(10) Use $\cos^3\alpha = \frac{1}{4}(\cos 3\alpha + 3\cos\alpha)$.

(11) Use $\sin^4\alpha = \frac{1}{8}\cos 4\alpha - \frac{1}{2}\cos 2\alpha + \frac{3}{8}$.

(12) $\theta = \dfrac{2r\pi}{n}$, or $\dfrac{2}{n+1}(r\pi + \frac{1}{2}\pi)$ where r is any integer.

(13) 0 and $\frac{3}{8}n$.

(16) $\frac{1}{2}n(n+1)$ (17) $\frac{1}{4}n^2(n+1)^2$. (18) $\frac{1}{3}n(n+1)(n+2)$.

(19) Write $\delta + \pi$ for δ (20) Write $\delta + \pi$ for δ

XXVIII.

(1) $\operatorname{cosec}\theta\,\{\tan(n+1)\theta - \tan\theta\}$.

(2) $\operatorname{cosec}\theta\,\{\cot\theta - \cot(n+1)\theta\}$

(3) $\sec\theta\,\{\tan(n+1)(\theta - \frac{1}{2}\pi) + \cot\theta\}$, this may be proved by putting $\theta - \frac{1}{2}\pi$ for θ in Ex. (1).

(4) Each term in this series is one-half the corresponding term in Ex. (1).

(5) $\frac{1}{4}\operatorname{cosec}\frac{1}{2}\alpha\,\{\sec\frac{1}{2}(2n+1)\alpha - \sec\frac{1}{2}\alpha\}$

(6) $\frac{1}{4}\operatorname{cosec}\frac{1}{2}\alpha\,\{\operatorname{cosec}\frac{1}{2}\alpha - \operatorname{cosec}\frac{1}{2}(2n+1)\alpha\}$.

(7) $\frac{1}{2}\sin 2\theta - \dfrac{1}{2^{n+1}}\sin 2^{n+1}\theta$.

(8) $\frac{1}{2}\sin 2\theta - (-1)^n\dfrac{1}{2^{n+1}}\sin 2^{n+1}\theta$.

(9) $\frac{1}{4}\sin 2\theta - (-1)^n\, 2^{n-2}\sin\dfrac{\theta}{2^{n-1}}$.

(10) The result is similar to that in Ex (5)

(11) $\frac{1}{2}\sec\alpha\,\{\operatorname{cosec}\alpha - (-1)^n\operatorname{cosec}(2n+1)\alpha\}$.

(12) $\tan^{-1}(n+1) - \tan^{-1}1$.

(13) $\tan^{-1}(n+1)\alpha - \tan^{-1}\alpha$.

(14) $\tan^{-1}(n+1)^2\alpha^2 - \tan^{-1}\alpha^2$.

(15) $\cot\theta - 2^n\cot 2^n\theta$.

(16) The series reduces to [See Ex. 5, p. 80]

$$2(\operatorname{cosec}2\alpha + \operatorname{cosec}4\alpha + \operatorname{cosec}8\alpha + \,..\,\text{to } n \text{ terms}).$$

(17) $\frac{1}{2}\{\cos 2\theta - \cos 2^{n+1}\theta\}$. (18) $\frac{1}{2}\left\{\cos\dfrac{\theta}{2^{n-2}} - \cos 4\theta\right\}$.

(19) $\frac{1}{2}\operatorname{cosec}^2\dfrac{\theta}{2} - 2^{n-1}\operatorname{cosec}^2 2^{n-1}\theta$ (20) $\dfrac{1}{2^n}\operatorname{cosec}^2\dfrac{\theta}{2^n} - \operatorname{cosec}^2\theta$

(21) $\cot\dfrac{\alpha}{2^n} - \cot\alpha$.

(22) The series reduces to [Ex. 5, p. 80]

$$\sin\alpha\,(\operatorname{cosec}2\alpha + \operatorname{cosec}4\alpha + \operatorname{cosec}8\alpha + \ \text{to } n \text{ terms})$$

(23) $1 - \dfrac{1}{n+1}$. (24) $2^{2n} - 1$.

XXIX.

(1) $\dfrac{\sin\alpha}{1-2x\cos\alpha+x^2}$ (2) $\dfrac{\cos\alpha-x\cos(\alpha-\beta)}{1-2x\cos\beta+x^2}$.

(3) $\dfrac{\sin\alpha-\cos\alpha\sin(\alpha-\beta)}{1-2\cos\alpha\cos\beta+\cos^2\alpha}$. (4) $\dfrac{\cos\alpha-\sin\alpha\cos(\alpha-\beta)}{1-\sin\alpha\cos\beta+\sin^2\alpha}$.

(5) $e^{\cos\alpha}\sin(\sin\alpha)$ (6) $e^{x\cos\alpha}\sin(x\sin\alpha)$

(7) $e^{-\cos\alpha\cos\beta}\cos(\cos\alpha\sin\beta)$.

(8) $e^{-\cos\alpha\cos\beta}\sin(\alpha-\cos\alpha\sin\beta)$ (9) $\frac{1}{2}\alpha$.

(10) $-\log(2\sin\frac{1}{2}\alpha)$ (11) $\frac{1}{2}\log(1+2\sin\alpha\cos\beta+\sin^2\alpha)$.

(12) $-\frac{1}{2}\log(1-2\cos\theta\cos\phi+\cos^2\theta)$.

(13) $\dfrac{\cos\alpha-x\cos(\alpha-\beta)-x^n\cos(\alpha+n\beta)+x^{n+1}\cos\{\alpha+(n-1)\beta\}}{1-2x\cos\beta+x^2}$.

(14)
$$\frac{x\sin\alpha+x^2\sin(\alpha-\beta)+(-x)^{n+1}\sin(\alpha+n\beta)-(-x)^{n+2}\sin\{\alpha+(n-1)\beta\}}{1+2x\cos\beta+x^2}.$$

(15) $2^n\cos^n\frac{1}{2}\alpha\cos\frac{1}{2}n\alpha$

(16) $r^n\sin(\alpha+n\phi)$, when $r^2=1+2x\cos\beta+x^2$, and

$$\tan\phi=\frac{x\sin\beta}{1+x\cos\beta}.$$

(17) $\dfrac{1-\cos\alpha\cos\beta-\cos^n\alpha\{\cos n\beta-\cos\alpha\cos(n-1)\beta\}}{1-2\cos\alpha\cos\beta+\cos^2\alpha}$.

(18) $\dfrac{\sin\alpha\{1-\sin(\alpha-\beta)\}-\sin^n\alpha\{\sin(\alpha+n\beta)-\sin\alpha\sin(\alpha+\overline{n-1}\beta)\}}{1-2\sin\alpha\cos\beta+\sin^2\alpha}$.

(19) $\dfrac{(n+1)\sin n\alpha-n\sin(n+1)\alpha}{2(1-\cos\alpha)}$.

(20) $\dfrac{n^2\{\cos n\alpha-\cos(n+1)\alpha\}+2n\cos n\alpha}{2(1-\cos\alpha)}-\dfrac{\sin\alpha\sin n\alpha}{2(1-\cos\alpha)^2}$.

(21) $r^{\frac{p}{q}}\sin\dfrac{p\phi}{q}$, where $r^2=1+2\cos\alpha\cos\beta+\cos^2\alpha$, and

$$\tan\phi=\frac{\cos\alpha\sin\beta}{1+\cos\alpha\cos\beta}.$$

(22) The sum of $x\cos\theta-\frac{1}{3}x^3\cos 3\theta+\text{etc}=\frac{1}{2}\tan^{-1}\dfrac{2x\cos\theta}{1-x^2}$, $\therefore$ the required sum $=\frac{1}{4}(2n+1)\pi$.

(23) $e^{e^{\cos a}\cos(\sin a)}\cos\{e^{\cos a}\sin(\sin a)\}$.

(24) $\frac{1}{2}\tan^{-1}\frac{\cos y}{\sinh x}$. (25) $\tan^{-1}\frac{e^x\sin x}{1+e^x\cos x}$.

XXX.

(1) $\operatorname{cosec}\phi\,\{\sin\phi+a\sin 2\phi+a^2\sin 3\phi+\ \}$ by the method of Art. 77.

(2) $\sin a+x\sin 2a+x^2\sin 3a+\ldots$

(3) $1-x\cos\theta+x^2\cos 2\theta-x^3\cos 3\theta+\text{etc.}$

(4) $-2\{x\cos a+\frac{1}{2}x^2\cos 2a+\frac{1}{3}x^3\cos 3a+\ \}$

(5) $\sin a+x\sin(a+\beta)+x^2\sin(a+2\beta)+$

(6) $1+x\cos a+\frac{x^2\cos 2a}{1\,.\,2}+\frac{x^3\cos 3a}{1\,.\,2\,.\,3}+$

(7) $1+rx\cos\left(\tan^{-1}\frac{b}{a}\right)+\frac{r^2x^2}{1\,.\,2}\cos 2\left(\tan^{-1}\frac{b}{a}\right)+$
where $r^2=a^2+b^2$

(8) The coefficient of x^n is $\frac{2(a^2+b^2)^{\frac{n}{2}}}{\lfloor n}\cos\frac{n\pi}{4}\cos\left\{\frac{n}{2}\tan^{-1}\frac{a^2-b^2}{2ab}\right\}$.

(9) $x\sin a+\frac{x^2}{1\,.\,2}\sin 2a+\frac{x^3}{1\,.\,2\,.\,3}\sin 3a+$

(10) $\cos a+x\cos(a+\beta)+\frac{x^2}{1\,.\,2}\cos(a+2\beta)+$

(11) $\sin a+x\sin(a+\beta)+\frac{x^2}{1\,.\,2}\sin(a+2\beta)+\frac{x^3}{1\,.\,2\,.\,3}\sin(a+3\beta)+$

(13) Here $\frac{e^{i\phi}-e^{-i\phi}}{e^{i\phi}+e^{-i\phi}}=n\frac{e^{i\theta}-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$

$\therefore\ e^{2i\phi}=e^{2i\theta}\frac{1+re^{-2i\theta}}{1+re^{2i\theta}}$ where $r=\frac{1-n}{1+n}$.

$\therefore\ 2i\phi+2im\pi=2i\theta+\log(1+re^{-2i\theta})-\log(1+re^{2i\theta})$

$\therefore\ \phi+m\pi=\theta-r\sin 2\theta+\frac{1}{2}r^2\sin 4\theta-\ $.

(14) $\sec^n\theta=2^n\{\cos n\theta-n\cos(n+2)\theta+\frac{n(n+1)}{1\,.\,2}\cos(n+4)\theta-\ \}$

(15) $\cos na\cos^n a=1-\frac{n(n+1)}{1\,.\,2}\tan^2 a+\frac{n(n+1)(n+2)(n+3)}{1\,.\,2\,.\,3\,.\,4}\tan^4 a-$

(16) These series are recurring series

(i) $\dfrac{1-3\cos\theta}{1-6\cos\theta+9}+\dfrac{3(1-2\cos\theta)}{1-4\cos\theta+4}$;

(ii) $\dfrac{1}{3}\left\{\dfrac{x\sin\theta}{1-2x\cos\theta+x^2}+\dfrac{8x\sin\theta}{1-8x\cos\theta+16x^2}\right\}+1$

XXXI

(1) Use the second of the identities in Art. 89.

(5) Put $\frac{1}{2}\pi$ for θ in the expression of $\sin\theta$ in factors. This expression is known as Wallis's expression for π.

(15) Put ix for θ in both sides of the identity.

(17) $\dfrac{\sin(\theta+\alpha)}{\sin\alpha}=\cos\theta+\cot\alpha\sin\theta$, use this transformation in Ex. (8), expand $\cos\theta+\cot\alpha\sin\theta$ in ascending powers of θ by Art. 41 and equate the coefficients of θ on each side.

(19) This result may be deduced from Ex. (13)

(20) Put $\frac{1}{2}\pi-\alpha$ for α in (19).

XXXIII.

(1) (i) 8·4832462. (ii) 8·0563377. (iii) 8·4834473.

(2) (i) $1^0 44' 36''\cdot 8$ (ii) $55' 37''\cdot 4$. (iii) $1^0 44' 36''\cdot 8$.

XLIX.

(1) $-1\cdot708\{1\pm2\cdot0035\}$ (2) $3\cdot9705\{1\pm\cdot91032\}$

L.

(1) (i) $2\cos40^0,\ 2\cos160^0,\ 2\cos80^0$.

(ii) $\sqrt{2}\cos15^0,\ \sqrt{2}\cos135^0,\ \sqrt{2}\cos105^0$.

(iii) $2\cos40^0+1,\ 2\cos160^0+1,\ 2\cos80^0+1$.

(iv) $2\sqrt{2}\cos45^0-4,\ 2\sqrt{2}\cos165^0-4,\ 2\sqrt{2}\cos75^0-4$.

(v) $4\cos10^0-\sqrt{3},\ 4\cos130^0-\sqrt{3},\ 4\cos110^0-\sqrt{3}$.

(2) (i) $3a=90^0-1^0 22'$. (ii) $3a=65^0 41' 53''$. (iii) $3a=74^0 55' 47''$.

(3) (i) Find α and β such that $a=\tan\alpha$, $b=\tan\beta$, then
$$a\pm b=\sin(\alpha\pm\beta)\sec\alpha\sec\beta.$$

(ii) $a\cos\theta\pm b\sin\theta=a\cos(\theta\mp\alpha)\sec\alpha$, where $\tan\alpha=\frac{b}{a}$

(iii) $4\sin\frac{1}{2}(B+C)\sin\frac{1}{2}(C+A)\sin\frac{1}{2}(A+B)$

(iv) $4\sin(\theta-\alpha)\sin(m\theta-\alpha)\cos(\theta-m\theta)$.

(v) $4R\sin A\sin B\sin C$.

LI.

(2) Divide the first equation by a, the second by b, square both sides and add.

(3) The second equation gives $\cos^2\alpha=\frac{y-b}{a-b}$.

(6) $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2-3\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)-\frac{2x}{a}=0$

(7) $b^2=c^2-2ac\cos 2\phi+a^2$.

(9) $\cos 4\theta=\frac{1}{3}(2m^2-5)$, $\sin(\alpha-4\theta)=\frac{1}{3}\sin\alpha$, and so on

LIII.

(85) $\frac{1}{16}\pi^3$.

(91) $\sin\theta$ is greater than $\theta-\frac{1}{6}\theta$, when θ^3 is less than $\frac{1}{2}\pi$.

$\cos\theta$ is less than $1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4$. [Art. 44, 45]

Therefore $\tan\theta$ is greater than $\frac{\theta-\frac{1}{6}\theta^3}{1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4}$.

That is, $\tan\theta$ is greater than $\theta+\frac{1}{3}\theta^3+\frac{\frac{1}{8}\theta^5(1-\frac{1}{5}\theta^2)}{1-\frac{1}{2}\theta^2+\frac{1}{24}\theta^4}$, and if θ is less than $\frac{1}{2}\pi$ the last fraction is positive.

(96) $\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{(n+1)(n+2)}$, $\tan^{-1}\frac{1}{2}$.

(98) $\frac{1}{2\sin\theta\cos n\theta\cos(n+1)\theta}-\frac{1}{\sin 2\theta}$.

(101) $\frac{1}{4}\{\cot\frac{1}{2}\theta-3^n\cot\frac{1}{2}(3^n\theta)\}$.

(104) See Ex. 100. (110) $2\cos\theta\, e^{\theta\cot\theta}$.

(111) (i) $1+\frac{e\sin\alpha-e^n\sin n\alpha+e^{n+1}\sin(n-1)\alpha}{1-2e\cos\alpha+e^2}$.

(ii) $\tan\theta-\tan\frac{\theta}{2^n}$. (112) $\frac{-\sin\frac{1}{2}n(\pi-\alpha)}{2^n\sin^n\frac{1}{2}\alpha}$.

ANSWERS TO EXAMINATION PAPERS.

I.

5. $\cdot 469296$; $71^\circ 44' 29''$, $48^\circ 15' 31''$. 6 $\dfrac{l^2}{\cot^2 18^0 + \cot^2 30''}$.

II.

1. $130^0 - \dfrac{270^0}{\pi}$, $\frac{3}{4}\pi - \frac{1}{2}$ 2. 120^0, 240^0.

3. $n\pi + \frac{1}{6}\pi$, $n\pi + \frac{1}{4}\pi$. 7. $1 - \log 2$, $3 \log 2 - 3$, $\frac{3}{2} \log 2 - 1$.

10. $74^0 50' 38''$, $50^0 32' 58''$. 11. 5 ft. 4 in., $6\frac{4}{5}$ ft.

III.

7. $C = 18^0$, $b = \operatorname{cosec} 18^0$, $a = \cot 18^0 - 1$ 8. $78^0 27' 27''$.

11. 156·4, 556·4.

VI.

2. $2 \cos (\beta - \gamma) \cos (\gamma - \alpha) \cos (\alpha - \beta)$.

4. $\dfrac{x}{b \sin C} - \dfrac{y \cot C}{b}$, $\dfrac{y}{a \sin C} - \dfrac{x}{a} \cot C$.

VII.

2. $\tan \theta = \pm \dfrac{m^2 - n^2}{2mn}$, $\cos 2\theta = \dfrac{m^4 + n^4 - 6m^2n^2}{(m^2 + n^2)^2}$.

3. $\cos (\theta + \frac{1}{3}\pi) = 1$. 5. $AD = \dfrac{CD \sin ABD \cos ABC}{\sin (ABC - ABD)}$.

8. $B - C = 90^0$. 10 (2) $\frac{1}{2}\theta - \frac{1}{2}(2r + 1)\pi$.

VIII.

1. $\frac{2}{3} - \dfrac{1}{\pi}$. 3. $A + B + C = (n + \frac{1}{2})\pi$. 8. $-\frac{1}{8}$.

10. (1) $\dfrac{\sin \frac{1}{2}(\alpha + 3n\alpha) \sin \frac{3}{2} n\alpha}{\sin \frac{3}{2}\alpha}$; (2) See XXIX. (16).

IX.

1. $\frac{108}{\pi} = 34''.3$ 8. See Art. 125. 10. See Ex. XXIX. 19.

X.

11. (i) See Ex. XXVIII. (22). (ii) See Art. 32.

(iii) See Ex. XXX. (18).

XI.

1. $\frac{A^0}{l} + \frac{C^0}{n} = \frac{2B^0}{m}$, $\frac{A^0 C^0}{pq} = \left(\frac{B^0}{r}\right)^2$, $A^0 + B^0 + C^0 = 180^0$.

2. $\theta = 2\beta + 2n\pi$, $\phi = 2\alpha + 2m\pi$. 5. $\frac{1}{16}\pi^3$

11. (2) See XLIII. (96).

XII.

6. -1.

XIII.

1. $\sin(\frac{7}{2}\theta - \frac{1}{4}\pi)\cos(\frac{5}{2}\theta + \frac{1}{4}\pi)\cos\theta = 0$,

vii. $\cos 2n\theta = \left(1 - \frac{2n(2n-1)}{\lfloor 2}\tan^2\theta + \quad\right)(1 + \tan^2\theta)^n$

[Art. 19.]

XIV.

12. (2) $\frac{(2n+1)\sin n\alpha - (2n-1)\sin(n+1)\alpha - \sin\alpha}{2(1-\cos\alpha)}$.

XV.

9. See answer to Ex. XXV. (10).

10. $r\pi + \frac{1}{2}\pi - \{x^{-1} - \frac{1}{2}x^{-2}(\sin\alpha)^{-2}\sin 2\alpha + \frac{1}{3}x^{-3}(\sin\alpha)^{-3}\sin 3\alpha - \quad\}$.

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